Multiloop amplitudes of light-cone gauge superstring field theory in noncritical dimensions

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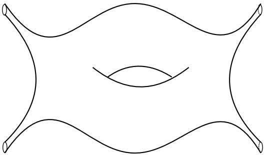
String Field Theory

- String field theory provides a nonperturbative formulation of string theory.
- It should reproduce the scattering amplitudes calculated by the first-quantized formalism.

$$A = \sum_{\text{worldsheet}}$$



In this talk

I would like to explain that

- Several bosonic string field theories are proved to reproduce the first-quantized results.
- So far, there are no such superstirng field theories. This is because there is still something not well-understood in the first-quantized formalism.
- The light-cone gauge superstring field theory can be proved to reproduce the first-quantized results by considering the dimensional regularization of the theory.

In collaboration with Y. Baba and K. Murakami

Outline

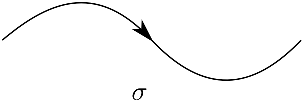
§1 Bosonic SFT vs. 1-st quantization

§1-1 Witten's SFT

Witten's cubic SFT (bosonic) (1986)

$$S = \int \left[\frac{1}{2} \Psi Q \Psi + \frac{g}{3} \Psi \cdot (\Psi * \Psi) \right]$$

- String field: $\Psi \left[X^{\mu}(\sigma), b(\sigma), c(\sigma) \right]$



$$0 \leq \sigma \leq \pi$$

Perturbation theory of bosonic strings

Taking the Siegel gauge $b_0\Psi = 0$,

- gauge fixed action

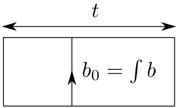
$$S = \int \left[\frac{1}{2}\Psi' c_0 L_0 \Psi' + \frac{g}{3}\Psi' \cdot (\Psi' * \Psi') \right]$$

- Feynman rule

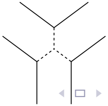
propagator

$\frac{b_0}{L_0} = b_0 \int_0^\infty dt e^{-tL_0}$

$\int_0^\infty dt$

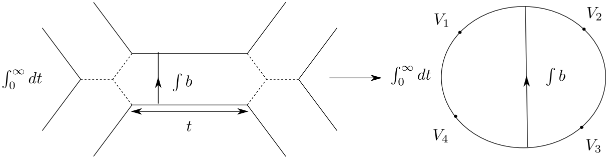


vertex



Feynman diagram

Four point tree amplitude



General amplitudes are expressed in the form

$$A_N = \sum_{\text{worldsheet}} \int \prod_\alpha dt_\alpha \left\langle V_1 \cdots V_N \prod_\alpha \int_{C_\alpha} b \right\rangle_{\text{worldsheet}}$$

with t_α: Feynman parameters

Amplitudes from the first-quantized formalism

▸ GO

$$A_N = \sum_{\text{worldsheet}} \int \frac{[dg_{mn} dX^\mu]}{\text{rep.} \times \text{Weyl}} e^{-I} V_1 \cdots V_N$$
$$= \sum_{\text{worldsheet}} \int \prod_\alpha dm_\alpha \, [dX^\mu dbdc] \, e^{-I_{\text{g.f.}}} V_1 \cdots V_N \prod_\alpha B_\alpha$$

- $\frac{\text{space of } g_{mn}}{\text{rep.} \times \text{Weyl}}$ =moduli space of worldsheet Riemann surface
- m_α: coordinates of the moduli space ▸ GO
- B_α: antighost insertions to soak up the zero modes:

$$B_\alpha = \int d^2\sigma \sqrt{g} \frac{\partial g_{mn}^{\text{rep.}}}{\partial m_\alpha} b^{mn}$$

Witten's SFT vs. 1-st quantization

$$\text{1-st} \quad : \quad \int \prod_\alpha dm_\alpha \, [dX^\mu dbdc] \, e^{-I_{\text{g.f.}}} V_1 \cdots V_N \prod_\alpha B_\alpha$$

$$= \int \prod_\alpha dm_\alpha \left\langle V_1 \cdots V_N \prod_\alpha B_\alpha \right\rangle$$

$$\updownarrow$$

$$\text{SFT} \qquad \int \prod_\alpha dt_\alpha \left\langle V_1 \cdots V_N \prod_\alpha \int_{C_\alpha} b \right\rangle$$

- The Feynman parameters t_α of SFT parametrize the moduli space of Riemann surfaces. (.....,Zwiebach 1991)
- $\int_{C_\alpha} b = \int d^2\sigma \sqrt{g} \frac{\partial g^{\text{rep.}}_{mn}}{\partial t_\alpha} b^{mn}$

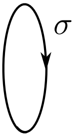
§1-2 Light-cone gauge SFT (closed)

Kaku-Kikkawa (1974)

$$S = \int \left[\frac{1}{2} \Phi \left(i\partial_\tau - \frac{L_0 + \tilde{L}_0 - 2}{\alpha} \right) \Phi + \frac{g}{3} \Phi \cdot (\Phi * \Phi) \right]$$

- string field:$\Phi \left[\tau, \alpha, X^i \left(\sigma \right) \right]$

$$\begin{aligned} \tau &= x^+ \\ \alpha &= 2p^+ \end{aligned}$$



$$0 \leq \sigma < 2\pi|\alpha|$$

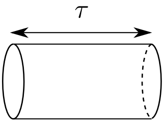
Perturbation theory

Feynmn rule

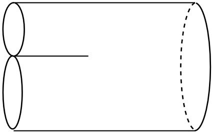
- propagator

$$\int_0^\infty d\tau e^{-i\tau \frac{L_0+\tilde{L}_0-2}{\alpha}}$$

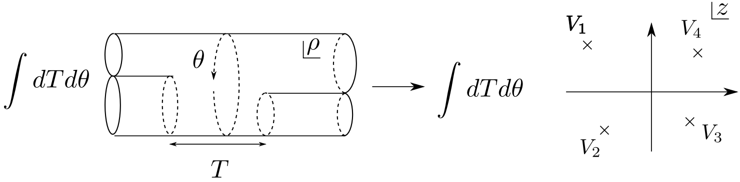
$$\int d\tau$$



- vertex



Feynman diagram



- The worldsheet theory is with $c = 24$, and we have an anomaly factor Γ.

$$A_N = \sum_{\text{worldsheet}} \int \prod_{\mathcal{I}} dt_{\mathcal{I}} \left\langle V_1^{\mathrm{LC}} \cdots V_N^{\mathrm{LC}} \right\rangle^{X^i} e^{-\Gamma}$$

Light-cone gauge SFT vs. 1-st quantization

$$\int \prod_{\mathcal{I}} dt_{\mathcal{I}} \left\langle V_1^{\mathrm{LC}} \cdots V_N^{\mathrm{LC}} \right\rangle^{X^i} e^{-\Gamma} \longleftrightarrow \int \prod_{\alpha} dm_{\alpha} \left\langle V_1 \cdots V_N \prod_{\alpha} B_{\alpha} \right\rangle$$

- $t_{\mathcal{I}}$'s cover the moduli space once. (Giddings-Wolpert)
- The integrands are also equal. (D'Hoker-Giddings)

The integrand

$$\prod_{\mathcal{I}} dt_{\mathcal{I}} \left\langle V_1^{\mathrm{LC}} \cdots V_N^{\mathrm{LC}} \right\rangle^{X^i} e^{-\Gamma} \longleftrightarrow \prod_{\alpha} dm_\alpha \left\langle V_1 \cdots V_N \prod_{\alpha} B_\alpha \right\rangle^{X^\mu, b, c}$$

$$e^{-\Gamma} \sim \left\langle c\bar{c}(z_1) \cdots c\bar{c}(z_N) \prod_{\mathcal{I}} B_{\mathcal{I}} \right\rangle^{X^\pm, b, c}$$

$$\left\langle V_1^{\mathrm{LC}} \cdots V_N^{\mathrm{LC}} \right\rangle^{X^i} \sim \left\langle V_1^{\mathrm{DDF}} \cdots V_N^{\mathrm{DDF}} \right\rangle^{X^\mu}$$

$$\left\langle V_1^{\mathrm{LC}} \cdots V_N^{\mathrm{LC}} \right\rangle^{X^i} e^{-\Gamma} = \left\langle \left(c\bar{c}V^{\mathrm{DDF}}\right)_1 \cdots \left(c\bar{c}V^{\mathrm{DDF}}\right)_N \prod_{\mathcal{I}} B_{\mathcal{I}} \right\rangle^{X^\mu, b, c}$$

GO

§1-3 Remarks

- Zwiebach's closed string field theory also reproduces the 1-st quantized results.
- The light-cone gauge open string field theory has not been proved to reproduce the 1-st quantized results.
- $\alpha = p^+$ HIKKO and the covariantized light-cone SFT (closed) reproduce the results of the light-cone gauge SFT and therefore the 1-st quantized results.

§2 Super SFT vs. 1-st quantization

§2-1 Witten's SFT for superstrings

Witten's cubic SFT for superstrings (1987)

$$S = \int \left[\frac{1}{2} \Psi Q \Psi + \frac{g}{3} \Psi \cdot X\left(\frac{\pi}{2}\right) (\Psi * \Psi) \right] + \text{fermions}$$

- $X(\sigma)$: picture changing operator

$$X(\sigma) = \delta(\beta) G(\sigma) + \cdots$$

$G(\sigma)$: worldsheet supercharge

$$G = \psi^{\mu} i \partial X_{\mu} + \text{ghost part}$$

Superstring perturbation theory

Taking the Siegel gauge $b_0\Psi = 0$

- gauge fixed action

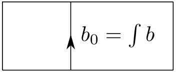
$$S = \int \left[\frac{1}{2}\Psi' c_0 L_0 \Psi' + \frac{g}{3}\Psi' \cdot X\left(\frac{\pi}{2}\right)(\Psi' * \Psi') \right]$$

- Feynman rule

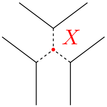
propagator

$\frac{b_0}{L_0} = b_0 \int_0^\infty dt e^{-tL_0}$

$\int_0^\infty dt$

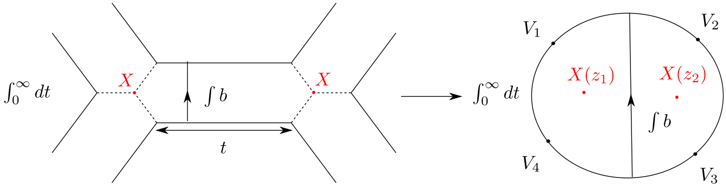


vertex



Superstring perturbation theory

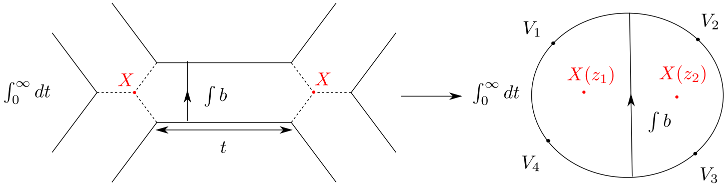
Four point tree amplitude



$$A = \int_0^\infty dt \left\langle V_1 \cdots V_4 X\left(z_1\left(t\right)\right) X\left(z_2\left(t\right)\right) \int b \right\rangle + \text{other channels}$$

Superstring perturbation theory

Four point tree amplitude



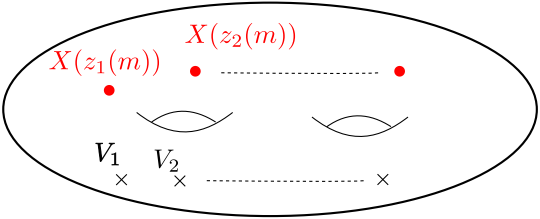
$$A = \int_0^\infty dt \left\langle V_1 \cdots V_4 X\left(z_1\left(t\right)\right) X\left(z_2\left(t\right)\right) \int b \right\rangle + \text{other channels}$$

The integral diverges because $z_1\left(0\right) = z_2\left(0\right)$ and

$$X\left(z_1\right) X\left(z_2\right) \sim \left(z_1 - z_2\right)^{-2} \times \text{regular operator } \left(z_1 \sim z_2\right)$$

Witten's SFT for superstrings

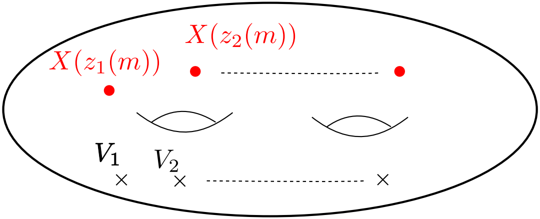
- The amplitudes can be given in the form

$\int \prod_\alpha dm_\alpha$ 

- The integral diverges when the picture changing operators collide. (contact term problem) (Wendt, 1987)
- Various ways to avoid the contact term problems are proposed. (modified cubic, Berkovits, ...)

§2-2 The 1-st quantized superstring theory

- Although they are divergent, the amplitudes from Witten's superstring field theory are of the form

$\int \prod_\alpha dm_\alpha$ 

- The 1-st quantized formalism yields the same form of the amplitudes.

Amplitudes from the first-quantized formalism

Martinec, Chaudhuri-Kawai-Tye GO

$$A = \sum_{\text{worldsheet}} \int \frac{[dg_{mn} d\chi_a dX^\mu d\psi^\mu]}{\text{superrep.} \times \text{superWeyl}} e^{-I} V_1 \cdots V_N$$
$$= \sum_{\text{worldsheet}} \int \prod_\alpha dm_\alpha \prod_\sigma d\eta_\sigma \, [dX^\mu d\psi^\mu dbdcd\beta d\gamma]$$
$$\times e^{-I_{\text{g.f.}}} V_1 \cdots V_N \prod_\alpha B_\alpha \prod \delta(\beta_\sigma)$$

- $\frac{\text{space of } g_{mn}, \chi_a}{\text{superrep.} \times \text{superWeyl}}$ =supermoduli space of superRiemann surface
- m_α,η_σ: coordinates of the supermoduli space GO
- $B_\alpha, \delta(\beta_\sigma)$: antighost insertions to soak up the zero modes

$$\beta_\sigma = \int d^2 z \frac{\partial \chi_{\bar{z}}^{\theta \text{rep.}}}{\partial \eta_\sigma} \beta$$

Picture changing operator

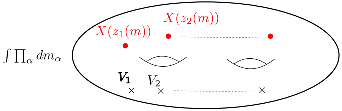
Verlinde-Verlinde

If one takes η_σ so that $\frac{\partial \chi_{\bar{z}}^{\theta \mathrm{rep.}}}{\partial \eta_\sigma} = \delta^2 (z - z_\sigma)$ and integrating over η_σ we get

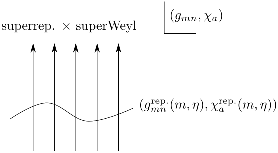
$$\begin{aligned} A &= \sum_{\text{worldsheet}} \int \prod_\alpha dm_\alpha \prod_\sigma d\eta_\sigma \left[dX^\mu d\psi^\mu dbdcd\beta d\gamma \right] \\ &\qquad \times e^{-I_{\mathrm{g.f.}}} V_1 \cdots V_N \prod_\alpha B_\alpha \prod_\sigma \delta(\beta_\sigma) \\ &= \sum_{\text{worldsheet}} \int \prod_\alpha dm_\alpha \left[dX^\mu d\psi^\mu dbdcd\beta d\gamma \right] \\ &\qquad \times e^{-I} V_1 \cdots V_N \prod_\alpha B_\alpha \prod_\sigma X(z_\sigma) \end{aligned}$$

Picture changing operators

- Taking $\frac{\partial \chi_{\bar{z}}^{\theta\text{rep.}}}{\partial \eta_\sigma} = \delta^2(z - z_\sigma)$ we get the amplitudes with picture changing operators inserted.

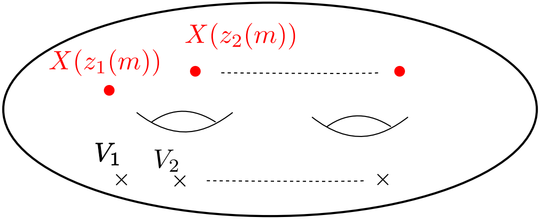


- We can freely take z_σ as long as $\frac{\partial \chi_{\bar{z}}^{\theta\text{rep.}}}{\partial \eta_\sigma}$ $(\sigma = 1, \cdots 2g - 2 + N)$ span the space transverse to the symmetry orbits. It is a "gauge choice".



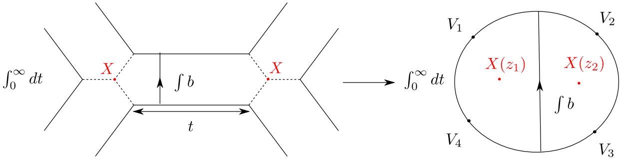
Amplitudes from the super SFT

$\int \prod_\alpha dm_\alpha$



- This would be correct, if $z_\sigma(m)$ corresponded to a good "gauge choice".

Contact term problem



$$A_4 = \int_0^\infty dt F(t)$$

$$F(t) = \left\langle V_1 \cdots V_4 X(z_1(t)) X(z_2(t)) \int b \right\rangle$$

- The amplitude diverges at $t = 0$, because $\frac{\partial \chi_{\bar{z}}^{\theta \text{rep.}}}{\partial \eta_1}, \frac{\partial \chi_{\bar{z}}^{\theta \text{rep.}}}{\partial \eta_2}$ do not span the two dimensional space transverse to the symmetry orbit. Namely it is a bad "gauge choice" at $t = 0$.

From the point of view of 1-st quantized formalism

Since the "gauge" we choose is not good at $t = 0$ in

$$A_4 = \int_0^\infty dt F(t)$$

$$F(t) = \left\langle V_1 \cdots V_4 X(z_1(t)) X(z_2(t)) \int b \right\rangle$$

why don't we take a different gauge for $t \sim 0$, namely a different way to place the picture changing operators:

$$A_4 = \int_a^\infty dt F(t) + \int_0^a dt F'(t) \; ?$$

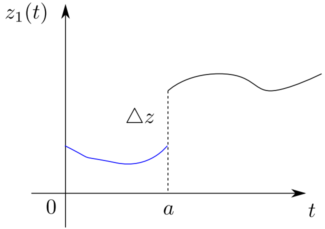
$$F'(t) = \left\langle V_1 \cdots V_4 X(z_1(t) + \Delta z) X(z_2(t)) \int b \right\rangle$$

In order to avoid divergence

$$A_4 = \int_a^\infty dt F(t) + \int_0^a dt F'(t) \; ?$$

$$F(t) = \left\langle V_1 \cdots V_4 X(z_1(t)) X(z_2(t)) \int b \right\rangle$$

$$F'(t) = \left\langle V_1 \cdots V_4 X(z_1(t) + \Delta z) X(z_2(t)) \int b \right\rangle$$



This does not work

$$A_4(a) = \int_a^\infty dt F(t) + \int_0^a dt F'(t) \ ?$$

$A_4(a)$ depends on how we choose a.

$$\partial_a A_4(a) = F'(a) - F(a) \neq 0$$

Since there is no canonical way to choose a, the result becomes ambiguous.

$F'(t) - F(t)$

$$F(t) = \left\langle V_1 \cdots V_4 X(z_1(t)) X(z_2(t)) \int b \right\rangle$$
$$F'(t) = \left\langle V_1 \cdots V_4 X(z_1(t) + \Delta z) X(z_2(t)) \int b \right\rangle$$

Since

$$X(z_1(t) + c) - X(z_1(t)) = \{Q, \xi(z_1(t) + \Delta z) - \xi(z_1(t))\}$$

we get

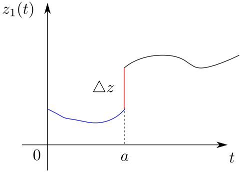
$$\begin{aligned} F'(t) - F(t) &= \left\langle V_1 \cdots V_4 \{Q, \chi(t)\} X(z_2(t)) \int b \right\rangle \\ &= \partial_t \langle V_1 \cdots V_4 (\xi(z_1(t) + \Delta z) - \xi(z_1(t))) X(z_2(t)) \rangle \\ &\equiv \partial_t f(t) \neq 0 \end{aligned}$$

The correct amplitude is given as

$$A_4 = \int_a^\infty dt F(t) + \int_0^a dt F'(t) - f(a)$$

- $\partial_a (A_4) = F'(a) - F(a) - \partial_a f(a) = 0$
- $f(a)$ comes from the "vertical segment" (Saroja-Sen 1992, Sen 2014)

$$f(a) = \int_{z_1(a)}^{z_1(a)+\Delta z} dz \langle V_1 \cdots V_4 \partial\xi(z) X(z_2(t)) \rangle$$



General amplitudes in the first quantized formalism

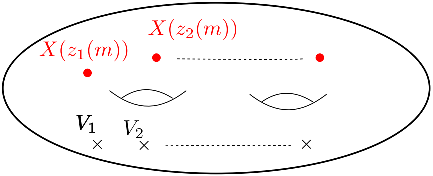
They suffer from divergences coming from the bad "gauge choice".

$$\left\langle \prod_i \delta(\beta)(z_i) \prod_r \delta(\gamma)(Z_r) \right\rangle$$

$$\propto \frac{1}{\vartheta[\alpha]\left(\sum z_i - \sum Z_r - 2\triangle\right)} \cdot \frac{\prod_{i,r} E(z_i, Z_r)}{\prod_{i>j} E(z_i, z_j) \prod_{r>s} E(Z_r, Z_s)} \cdot \frac{\prod_r \sigma(Z_r)^2}{\prod_i \sigma(z_i)^2}$$

$$\begin{cases} z_i = z_j & \text{: contact term} \\ \vartheta[\alpha]\left(\sum z_i - \sum Z_r - 2\triangle\right) = 0 & \text{: spurious poles} \end{cases}$$

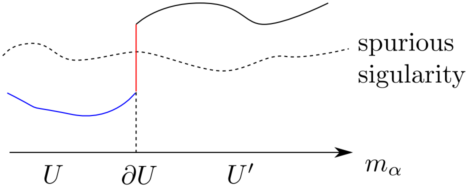
$\int \prod_\alpha dm_\alpha$



General amplitudes

In order to avoid the singularities, we divide the moduli space into patches and

$$A = \sum_{U} \int_{U} \prod_{\alpha} dm_{\alpha} F(m) + \sum_{\partial U} \int_{\partial U} f_{\partial U}$$



- The spurious singularities are of codimension 2 and we need the discontinuities.

General amplitudes

$$A = \sum_U \int_U \prod_\alpha dm_\alpha F(m) + \sum_{\partial U} \int_{\partial U} f_{\partial U}$$

This expression is not useful in calculating the amplitudes. It is better if we do not have the second term.

- We can have an expression without the second term if the supermoduli space is projected/split.
- For higher genera, the supermoduli space is not holomorphically projected/split. (Donagi-Witten 2013)

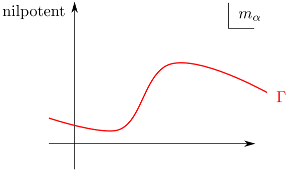
GO

Witten's prescription

The expression as an integral over the supermoduli space is more illuminating

$$A = \int_{\Gamma} \prod_{\alpha} dm_{\alpha} \prod_{\sigma} d\eta_{\sigma} \Lambda\left(m, \eta\right)$$

- Γ is a contour of m_α which can have a nilpotent part.
- Γ is taken to be any contour because Λ is analytic in m_α, if it behaves well at infinity.



§2-3 Super SFT vs. 1-st quantization

The first quantized formalism yields

$$\begin{aligned} A &= \int_\Gamma \prod_\alpha dm_\alpha \prod_\sigma d\eta_\sigma \Lambda(m,\eta) \\ &= \sum_U \int_U \prod_\alpha dm_\alpha F(m) + \sum_{\partial U} \int_{\partial U} f_{\partial U} \end{aligned}$$

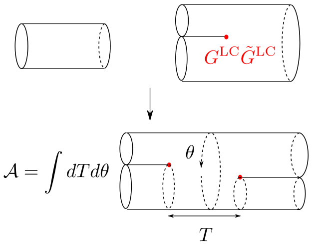
- Neither of these are useful for practical calculations.
- A superstring field theory should provide a systematic rule to yield either of these
- or more enlightening expression.

§3 Dimensional regularization of the light-cone gauge SFT

§3-1 Light-cone gauge super SFT

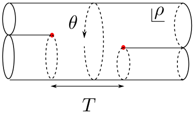
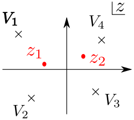
$\tau = x^+$ (Mandelstam, S.J. Sin)

$$S = \int \left[\frac{1}{2} \Phi \cdot \left(i\partial_\tau - \frac{L_0 + \tilde{L}_0 - 1}{\alpha} \right) \Phi + \frac{g}{3} \Phi \cdot (\Phi * \Phi) \right]$$



Contact term problem (Mandelstam, Klinkhamer-Greensite)

Light-cone gauge super SFT

$\int dT d\theta$  $\longrightarrow \int dT d\theta$ 

As in the bosonic theory

$$A = \int \prod_{\mathcal{I}} dt_{\mathcal{I}} \left\langle \prod_{I=1}^{2g-2+N} \left| \left(\partial^2 \rho\right)^{-\frac{3}{4}} G^{\mathrm{LC}}(z_I) \right|^2 \prod_{r=1}^{N} V_r^{\mathrm{LC}} \right\rangle^{X^i,\psi^i} e^{-\frac{1}{2}\Gamma}$$

$$= \int \prod_{\mathcal{I}} dt_{\mathcal{I}} \left\langle \prod_{\mathcal{I}} \oint \left(\mu_{\mathcal{I}} b + \bar{\mu}_{\mathcal{I}} \bar{b}\right) \prod_{I=1}^{2g-2+N} X\bar{X}(z_I, \bar{z}_I) \prod_{r=1}^{N} V_r^{\mathrm{conf.}} \right\rangle^{X^\mu,\psi^\mu,\mathrm{ghosts}}$$

The contact term problem has the same origin as the one in the Witten's super SFT.

No spurious poles

$$
\begin{aligned}
A &= \int \prod_{\mathcal{I}} dt_{\mathcal{I}} \left\langle \prod_{I=1}^{2g-2+N} \left| \left(\partial^2 \rho\right)^{-\frac{3}{4}} G^{\mathrm{LC}}\left(z_I\right) \right|^2 \prod_{r=1}^{N} V_r^{\mathrm{LC}} \right\rangle^{X^i, \psi^i} e^{-\frac{1}{2}\Gamma} \\
&= \int \prod_{\mathcal{I}} dt_{\mathcal{I}} \left\langle \prod_{\mathcal{I}} \oint \left(\mu_{\mathcal{I}} b + \bar{\mu}_{\mathcal{I}} \bar{b}\right) \prod_{I=1}^{2g-2+N} X\bar{X}\left(z_I, \bar{z}_I\right) \prod_{r=1}^{N} V_r^{\mathrm{conf.}} \right\rangle^{X^\mu, \psi^\mu, \mathrm{ghosts}}
\end{aligned}
$$

- The first line involves no $\beta\gamma$ system.
- In the second line, $\vartheta[\alpha](0)$ which comes from the $\psi^{\pm}$ cancels

$$
\frac{1}{\vartheta[\alpha]\left(\sum z_i - \sum Z_r - 2\triangle\right)} = \frac{1}{\vartheta[\alpha](0)}
$$

Only the contact term problem

§3-2 Dimensional regularization

Light-cone gauge SFT can be formulated in any d

$$S = \int \left[\frac{1}{2} \Phi \cdot \left(i\partial_\tau - \frac{L_0 + \tilde{L}_0 - \frac{d-2}{8}}{\alpha} \right) \Phi + \frac{g}{3} \Phi \cdot (\Phi * \Phi) \right]$$

- LC gauge SFT is a completely gauge fixed theory.
- The Lorentz invariance is broken.

Dimensional regularization

Even for $d \neq 10$, following the same procedure as that in the critical case

$$A = \int \prod_{\mathcal{I}} dt_{\mathcal{I}} \left\langle \prod_{I=1}^{2g-2+N} \left| (\partial^2 \rho)^{-\frac{3}{4}} G^{\mathrm{LC}}(z_I) \right|^2 \prod_{r=1}^{N} V_r^{\mathrm{LC}} \right\rangle^{X^i, \psi^i} e^{-\frac{d-2}{16}\Gamma}$$

$$= \int \prod_{\mathcal{I}} dt_{\mathcal{I}} \left\langle \prod_{\mathcal{I}} \oint (\mu_{\mathcal{I}} b + \bar{\mu}_{\mathcal{I}} \bar{b}) \prod_{I=1}^{2g-2+N} X\bar{X}(z_I, \bar{z}_I) \prod_{r=1}^{N} V_r^{\mathrm{conf.}} \right\rangle^{X^\mu, \psi^\mu, \mathrm{ghosts}}$$

but with a nontrivial CFT for $X^{\pm}$ ($X^{\pm}$ CFT).

- The worldsheet theory becomes BRST invariant

$$\begin{array}{ccccccccc} & & X^{\pm} & & X^i & & b,c & & \\ \hat{c} & = & 12-d & + & d-2 & - & 10 & = & 0 \end{array}$$

In the second-quantized language, DR is a gauge invariant regularization.

$X^{\pm}$ CFT

$$S_{X^{\pm}} = -\frac{1}{2\pi}\int d^2z d\theta d\bar{\theta}\left(\bar{D}X^{+}DX^{-}+\bar{D}X^{-}DX^{+}\right)+\frac{d-10}{8}\Gamma_{\mathrm{super}}[\Phi]$$

$$X^{\pm} \equiv x^{\pm}+i\theta\psi^{\pm}+i\bar{\theta}\tilde{\psi}^{\pm}+i\theta\bar{\theta}F^{\pm}$$

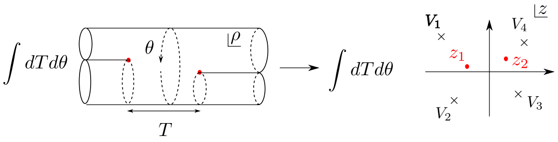
$$\Gamma_{\mathrm{super}}[\Phi] = -\frac{1}{2\pi}\int d^2z d\theta d\bar{\theta}\left(\bar{D}\Phi D\Phi+\theta\bar{\theta}\hat{g}_{z\bar{z}}\hat{R}\Phi\right)$$

$$\Phi \equiv \ln\left(\left(D\Theta^{+}\right)^2\left(\bar{D}\tilde{\Theta}^{+}\right)^2\right)-\ln\hat{g}_{z\bar{z}}$$

$$\Theta^{+} \equiv \frac{DX^{+}}{(\partial X^{+})^{\frac{1}{2}}}$$

- This theory can be formulated for $\langle\partial_m X^{+}\rangle \neq 0$
- It is a superconformal field theory with $\hat{c} = 12-d$.

Dimensional regularization



$$A = \int \prod_{\mathcal{I}} dt_{\mathcal{I}} \left\langle \prod_{I=1}^{2g-2+N} \left| \left(\partial^2\rho\right)^{-\frac{3}{4}} G^{\mathrm{LC}}\left(z_I\right) \right|^2 \prod_{r=1}^{N} V_r^{\mathrm{LC}} \right\rangle^{X^i,\psi^i} e^{-\frac{d-2}{16}\Gamma}$$

$$= \int \prod_{\mathcal{I}} dt_{\mathcal{I}} \left\langle \prod_{\mathcal{I}} \oint \left(\mu_{\mathcal{I}} b + \bar{\mu}_{\mathcal{I}} \bar{b}\right) \prod_{I=1}^{2g-2+N} X\bar{X}\left(z_I, \bar{z}_I\right) \prod_{r=1}^{N} V_r^{\mathrm{conf.}} \right\rangle^{X^\mu,\psi^\mu,\mathrm{ghosts}}$$

$$e^{-\frac{d-2}{16}\Gamma} \sim \left|z_I - z_J\right|^{-\frac{d-2}{8}} \text{ for } \left|z_I - z_J\right| \sim 0$$

By taking d to be large and negative, the amplitudes do not diverge. GO

Dimensional regularization

- What matters is the Virasoro central charge $\hat{c}$ rather than the number of the spacetime coordinates

$$A = \int \prod_{\mathcal{I}} dt_{\mathcal{I}} \left\langle \prod_{I=1}^{2g-2+N} \left| (\partial^2 \rho)^{-\frac{3}{4}} G^{\mathrm{LC}}(z_I) \right|^2 \prod_{r=1}^{N} V_r^{\mathrm{LC}} \right\rangle^{X^i, \psi^i} e^{-\frac{\hat{c}-2}{16}\Gamma}$$

- In order to incorporate the spacetime fermions, one can take the worldsheet theory to be for example

$$\begin{array}{ccccccc} & X^i, \psi^i & & SU(2)\text{super WZW} \times 2M & & \left(\hat{b}, \hat{c}, \hat{\beta}, \hat{\gamma}\right) \times 3M \\ c = & 12 & + & \left(\frac{3k}{k+2} + \frac{3}{2}\right) \times 2M & + & (-3) \times 3M \end{array}$$

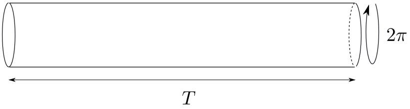
where $\left(\hat{b}, \hat{c}, \hat{\beta}, \hat{\gamma}\right)$ are of weight $\left(1, 0, \frac{1}{2}, \frac{1}{2}\right)$.

- We can deal with only the even spin structure amplitudes.

Remarks

- We can realize "SFT in fractional dimensions" and the regularization is not restricted to perturbation theory.
- The dimensional regularization works as a UV/IR regularization

$$S = \int \left[\frac{1}{2} \Phi \cdot \left(i\partial_\tau - \frac{L_0 + \tilde{L}_0 - \frac{\hat{c}-2}{8}}{\alpha} \right) \Phi + \frac{g}{3} \Phi \cdot (\Phi * \Phi) \right]$$



$$\sim e^{\frac{d-2}{8}T}$$

§3-3 Light-cone gauge super SFT vs. 1-st quantization

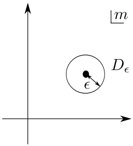
$$
\begin{aligned}
A^{\mathrm{LC}} &= \int \prod_{\mathcal{I}} dt_{\mathcal{I}} \left\langle \prod_{I=1}^{2g-2+N} \left| \left(\partial^2 \rho\right)^{-\frac{3}{4}} G^{\mathrm{LC}}\left(z_I\right) \right|^2 \prod_{r=1}^{N} V_r^{\mathrm{LC}} \right\rangle^{X^i, \psi^i} e^{-\frac{\hat{c}-2}{16}\Gamma} \\
&= \int \prod_{\mathcal{I}} dt_{\mathcal{I}} \left\langle \prod_{\mathcal{I}} \oint \left(\mu_{\mathcal{I}} b + \bar{\mu}_{\mathcal{I}} \bar{b}\right) \prod_{I=1}^{2g-2+N} X\bar{X}\left(z_I, \bar{z}_I\right) \prod_{r=1}^{N} V_r^{\mathrm{conf.}} \right\rangle^{X^\mu, \psi^\mu, \mathrm{ghosts}} \\
&= \int_{\mathcal{M}} d^2 m F\left(m, \bar{m}\right)
\end{aligned}
$$

- This expression is well-defined with $\hat{c}$ large and negative.
- We define the amplitudes for $d = 10$ by analytically continuing $\hat{c}$ to 10.

In order to get an expression valid for any $\hat{c}$

$$A^{\mathrm{LC}} = \int \prod_{\mathcal{I}} dt_{\mathcal{I}} \left\langle \prod_{\mathcal{I}} \oint \left(\mu_{\mathcal{I}} b + \bar{\mu}_{\mathcal{I}} \bar{b}\right) \prod_{I=1}^{2g-2+N} X\bar{X}\left(z_I, \bar{z}_I\right) \prod_{r=1}^{N} V_r^{\mathrm{conf.}} \right\rangle^{X^\mu, \psi^\mu, \mathrm{ghosts}}$$

- Taking a small neighborhood D_ϵ of the singularities, we get



$$A^{\mathrm{first}} = \int_{\mathcal{M}-\sum D_\epsilon} d^2 m F\left(m, \bar{m}\right) + \sum \int_{D_\epsilon} d^2 m F'\left(m, \bar{m}\right) + \sum \int_{\partial D_\epsilon} f$$

- This 1-st quantized expression is valid for any $\hat{c}$ and independent of ϵ.

Dimensional regularization

$$A^{\text{first}} = \int_{\mathcal{M}-\sum D_\epsilon} d^2 m F(m,\bar{m}) + \sum \int_{D_\epsilon} d^2 m F'(m,\bar{m}) + \sum \int_{\partial D_\epsilon} f$$

- For $\hat{c}$ large and negative,

$$\lim_{\epsilon\to 0} \int_{D_\epsilon} d^2 m F'(m,\bar{m}) = \lim_{\epsilon\to 0} \int_{\partial D_\epsilon} f = 0$$

and

$$A^{1-\text{st}} = \int_{\mathcal{M}} d^2 m F(m,\bar{m}) = A^{\text{LC}}$$

- The limit $\hat{c} \to 10$ in the light-cone results coincide with the 1-st quantized one.

§5 Conclusions and discussions

- There is still something not well-understood in the first quantized formalism of superstrings.
- Dimensional regularization of the light-cone gauge super SFT can be used to reproduce the results of the first quantized formalism.
- Supersymmetry breaking in superstring theory?
- Dimensional regularization in Witten's cubic SFT?
- Nonperturbative calculations by SFT?

First-quantized formalism

Worldsheet action

$$I = \frac{1}{8\pi} \int d^2\sigma \sqrt{g} g^{mn} \partial_m X^\mu \partial_n X_\mu$$

- reparametrization invariance: $\sigma^m \to \sigma^m + \epsilon^m(\sigma)$
- Weyl invariance: $g_{mn}(\sigma) \to e^{\varepsilon(\sigma)} g_{mn}(\sigma)$

Amplitude

$$A = \sum_{\text{worldsheet}} \int \frac{[dg_{mn} dX^\mu]}{\text{rep.} \times \text{Weyl}} e^{-I} V_1 \cdots V_N$$

- $V_i \ (i = 1, \cdots, N)$: vertex operators

$e^{-\Gamma}$

$$e^{-\Gamma} \propto \prod_r \left[e^{-2\mathrm{Re}\bar{N}_{00}^{rr}} \left(g^{\mathrm{A}}_{Z_r \bar{Z}_r} \right)^{-1} \alpha_r^{-2} \right] \prod_I \left[\left| \partial^2 \rho\left(z_I\right) \right|^{-1} g^{\mathrm{A}}_{z_I \bar{z}_I} \right]$$

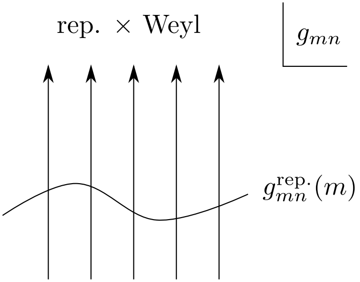
$g^{\mathrm{A}}_{z\bar{z}}$: Arakelov metric

$$\rho(z) = \sum_{r=1}^{N} \alpha_r \left[\ln E\left(z, Z_r\right) - 2\pi i \int^z \omega \left(\mathrm{Im}\Omega\right)^{-1} \mathrm{Im} \int^{Z_r} \omega \right]$$

$$\bar{N}_{00}^{rr} = \frac{1}{\alpha_r} \left[\rho\left(z_{I(r)}\right) - \lim_{z \to Z_r} \left(\rho(z) - \alpha_r \ln\left(z - Z_r\right) \right) \right]$$

BACK1 BACK2

First-quantized formalism



BACK

First-quantized formalism

Worldsheet action

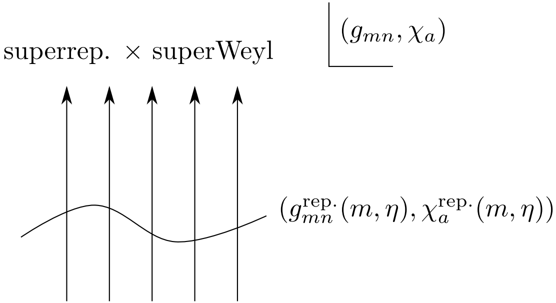
$$I = \frac{1}{8\pi}\int d^2\sigma\sqrt{g}\,[g^{mn}\partial_m X^\mu \partial_n X_\mu - i\psi^\mu\gamma^m\partial_m\psi_\mu \\ -\psi^\mu\gamma^a\gamma^m\chi_a\partial_m X_\mu + \frac{1}{4}\left(\psi^\mu\gamma^a\gamma^b\chi_a\right)\chi_b\psi_\mu\Big]$$

- χ_a: gravitino field on the worlsheet
- superreparametrization invariance and super Weyl invariance

Amplitude

$$A = \sum_{\text{worldsheet}} \int \frac{[dg_{mn}d\chi_a dX^\mu d\psi^\mu]}{\text{superrep.} \times \text{superWeyl}} e^{-I} V_1 \cdots V_N$$

First-quantized formalism



η_σ: odd moduli (Grassmann odd) BACK

Picture changing operator

A convenient choice is $\chi_{\bar{z}}^{(\sigma)\theta} = \delta^2(z - z_\sigma)$ and $\chi_{\bar{z}}^{\theta} = \sum_\sigma \eta_\sigma \delta^2(z - z_\sigma)$

$$\beta_\sigma = \int d^2 z \chi_{\bar{z}}^{(\sigma)\theta} \beta_{z\theta} = \beta(z_\sigma)$$

$$I_{\text{g.f.}} = \cdots + \int d^2 z \chi_{\bar{z}}^{\theta} G = I' + \sum_\sigma \eta_\sigma G(z_\sigma)$$

BACK

$$\int \prod_\alpha dm_\alpha \prod_\sigma d\eta_\sigma \left[dX^\mu d\psi^\mu dbdcd\beta d\gamma\right]$$

$$\times e^{-I_{\text{g.f.}}} V_1 \cdots V_N \prod_\alpha B_\alpha \prod_\sigma \delta(\beta_\sigma)$$

Picture changing operator

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BACK

$$\int \prod_\alpha dm_\alpha \prod_\sigma d\eta_\sigma \left[dX^\mu d\psi^\mu db dc d\beta d\gamma\right] \times e^{-I_{\text{g.f.}}} V_1 \cdots V_N \prod_\alpha B_\alpha \prod_\sigma \delta(\beta_\sigma)$$

$$\propto \int \prod_\alpha dm_\alpha \left[dX^\mu d\psi^\mu db dc d\beta d\gamma\right] \times e^{-I'} V_1 \cdots V_N \prod_\alpha B_\alpha \prod_\sigma \left(\delta(\beta) G + \cdots\right)(z_\sigma)$$

Picture changing operator

A convenient choice is $\chi_{\bar{z}}^{(\sigma)\theta} = \delta^2(z - z_\sigma)$ and $\chi_{\bar{z}}^{\theta} = \sum_\sigma \eta_\sigma \delta^2(z - z_\sigma)$

$$\beta_\sigma = \int d^2 z \chi_{\bar{z}}^{(\sigma)\theta} \beta_{z\theta} = \beta(z_\sigma)$$

$$I_{\text{g.f.}} = \cdots + \int d^2 z \chi_{\bar{z}}^{\theta} G = I' + \sum_\sigma \eta_\sigma G(z_\sigma)$$

BACK

$$\int \prod_\alpha dm_\alpha \prod_\sigma d\eta_\sigma \left[dX^\mu d\psi^\mu dbdcd\beta d\gamma\right] \times e^{-I_{\text{g.f.}}} V_1 \cdots V_N \prod_\alpha B_\alpha \prod_\sigma \delta(\beta_\sigma)$$

$$\propto \int \prod_\alpha dm_\alpha \left[dX^\mu d\psi^\mu dbdcd\beta d\gamma\right] \times e^{-I'} V_1 \cdots V_N \prod_\alpha B_\alpha \prod_\sigma X(z_\sigma)$$

Projected, split

the supermoduli space is covered by patches with the local coordinates which are related by the transformations of the form

$$m'_\alpha = f_\alpha(m) + \mathcal{O}(\eta^2)$$
$$\eta'_\sigma = \sum_{\sigma'} g_{\sigma\sigma'}(m)\,\eta_{\sigma'} + \mathcal{O}(\eta^3)$$

- If one can take the transformations of the form

$$m'_\alpha = f_\alpha(m)$$
$$\eta'_\sigma = \sum_{\sigma'} g_{\sigma\sigma'}(m)\,\eta_{\sigma'} + \mathcal{O}(\eta^3)$$

  the supermoduli space is projected.

Projected, split

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$$m'_\alpha = f_\alpha(m) + \mathcal{O}\left(\eta^2\right)$$
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Projected, split

the supermoduli space is covered by patches with the local coordinates which are related by the transformations of the form

$$
\begin{aligned}
m'_\alpha &= f_\alpha(m) + \mathcal{O}\left(\eta^2\right) \\
\eta'_\sigma &= \sum_{\sigma'} g_{\sigma\sigma'}(m)\,\eta_{\sigma'} + \mathcal{O}\left(\eta^3\right)
\end{aligned}
$$

- If the supermoduli space is projected

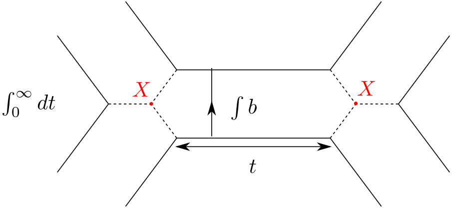
$$
\begin{aligned}
m'_\alpha &= f_\alpha(m) \\
\eta'_\sigma &= \sum_{\sigma'} g_{\sigma\sigma'}(m)\,\eta_{\sigma'} + \mathcal{O}\left(\eta^3\right)
\end{aligned}
$$

the amplitudes can be expressed

Backup

Contact term problem

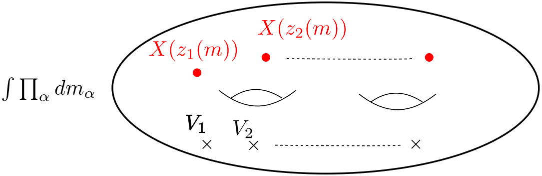
The superSFT yields obviously a wrong answer



- Even the four point tree amplitude is divergent, because the picture changing operators come close to each other
- This divergence spoils the gauge invariance of the theory.

This is called the contact term problem. (Wendt, 1987)

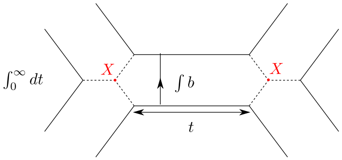
Problems



This way of calculation suffers from problems:

- spurious singularities
- total derivative ambiguity

SFT amplitude



$$\text{SFT} \quad : \quad \int_0^\infty dt \left\langle V_1 \cdots V_4 X\left(z_1\left(t\right)\right) X\left(z_2\left(t\right)\right) \int b \right\rangle$$

$$\updownarrow$$

$$\text{1-st} \quad : \quad \int \prod_\alpha dm_\alpha \left[dX^\mu d\psi^\mu dbdcd\beta d\gamma\right]$$

$$\times e^{-I} V_1 \cdots V_N \prod_\alpha B_\alpha \prod_\sigma X\left(z_\sigma\right)$$

The SFT amplitude corresponds to the specific choice

The way to deal with the contact term problem

In order to deal with the problem, modifications of Witten's action are proposed:

- modified cubic (Preitschopf-Thorn-Yost, Arefeva-Medvedev-Zubarev 1990)
  BRST invariance of multiloop amplitudes
  (Kohriki-Kishimoto-Kugo-Kunitomo-Murata 2011)
- Berkovits (1995)
  BRST invariance of tree amplitudes (Kroyter-Okawa-Schnabl-Torii-Zwiebach 2012)

These formulations take the string field to have pictures different from the canonical ones, it will need some work to relate these to the first-quantized results.