Energy from the gauge invariant observables

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§1 Introduction

A great variety of analytic classical solutions of Witten type OSFT has been discovered, especially since the discovery of Schnabl's tachyon vacuum solution.

Once a solution is found, there are two important gauge invariant quantities to be calculated.

- energy
- the gauge invariant observables

"Energy"

For a static solution $|\Psi\rangle$

$$\begin{aligned} E &\equiv -S \\ &= \frac{1}{g^2}\left[\frac{1}{2}\langle\Psi| Q |\Psi\rangle + \frac{1}{3}\langle\Psi|\Psi * \Psi\rangle\right] \\ &= E_\Psi - E_0 \end{aligned}$$

E_Ψ : "energy" for the vacuum corresponding to $|\Psi\rangle$

E_0 : "energy" for the perturbative vacuum

"the gauge invariant observables"

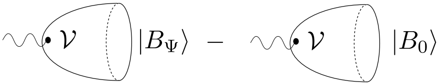
With an on-shell closed string vertex operator $\mathcal{V} = c\bar{c}V^{\mathrm{m}}$, one can construct

$$\langle I | \mathcal{V}(i) | \Psi \rangle$$

Hashimoto-Itzhaki, Gaiotto-Rastelli-Sen-Zwiebach

- $\langle I | \mathcal{V}(i) | \Psi \rangle$ corresponds to the difference of one point functions

$$-4\pi i \langle I | \mathcal{V}(i) | \Psi \rangle = \langle \mathcal{V} | c_0^- | B_\Psi \rangle - \langle \mathcal{V} | c_0^- | B_0 \rangle$$



Ellwood, Kiermaier-Okawa-Zwiebach

Energy and the gauge invariant observables

- Usually it is more difficult to calculate $E = \frac{1}{g^2}\left[\frac{1}{2}\langle\Psi|Q|\Psi\rangle + \frac{1}{3}\langle\Psi|\Psi * \Psi\rangle\right]$ compared to the gauge invariant observable $\langle I|\mathcal{V}(i)|\Psi\rangle$.
- For
$$\mathcal{V} \propto c\bar{c}\partial X^0 \bar{\partial} X^0 \,,$$
we expect that the gauge invariant observable is proportional to E.
- It will be useful to prove that the gauge invariant observable for such $\mathcal{V}$ yields E.

Energy from the gauge invariant observables

We would like to show

$$E = \frac{1}{g^2} \langle I | \mathcal{V}(i) | \Psi \rangle$$

for

$$\mathcal{V} = \frac{2}{\pi i} c\bar{c} \partial X^0 \bar{\partial} X^0$$

assuming that $|\Psi\rangle$ satisfies

- the equation of motion
- some regularity conditions

Takayuki Baba and N. I. arXiv:1208.6206

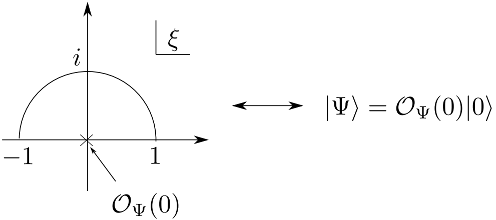
Outline

1. Introduction
2. A proof of $E = \frac{1}{g^2} \langle I | \mathcal{V}(i) | \Psi \rangle$
3. Solutions with K, B
4. Conclusions

§2 A proof of $E = \frac{1}{g^2} \langle I | \mathcal{V}(i) | \Psi \rangle$

String field

$$|\Psi\rangle = \mathcal{O}_\Psi(0)|0\rangle$$



Let us assume that $\mathcal{O}_\Psi$ is expressed in terms of really local operators located away from the arc $|\xi| = 1$.

A proof of $E = \frac{1}{g^2} \langle I | \mathcal{V}(i) | \Psi \rangle$

In order to prove $E = \frac{1}{g^2} \langle I | \mathcal{V}(i) | \Psi \rangle$, we consider for $h \ll 1$

$$S_h [|\Psi\rangle] \equiv -\frac{1}{g^2} \left[\frac{1}{2} \langle \Psi | Q | \Psi \rangle + \frac{1}{3} \langle \Psi | \Psi * \Psi \rangle + h \langle I | \mathcal{V}(i) | \Psi \rangle \right]$$

- $\mathcal{V} = \frac{2}{\pi i} c\bar{c}\partial X^0 \bar{\partial} X^0$ is a linear combination of graviton and dilaton vertex operators. c.f. Belopolsky-Zwiebach
- S_h should describe the string field theory in a constant metric and dilaton background. Zwiebach
- The constant metric can be gauged away and the effect of the constant dilaton is reduced to a rescaling of g.

Soft dilaton theorem

A "Soft dilaton theorem"

S_h can be shown to be equivalent to the original SFT action with a rescaling of g.

$$\begin{aligned} S_h[|\Psi\rangle] &= -\frac{1}{g^2}\left[\frac{1}{2}\langle\Psi|Q|\Psi\rangle + \frac{1}{3}\langle\Psi|\Psi*\Psi\rangle + h\langle I|\mathcal{V}(i)|\Psi\rangle\right] \\ &= -\frac{1+h}{g^2}\left[\frac{1}{2}\langle\Psi''|Q|\Psi''\rangle + \frac{1}{3}\langle\Psi''|\Psi''*\Psi''\rangle\right] + \mathcal{O}(h^2) \end{aligned}$$

$$|\Psi''\rangle = |\Psi\rangle + \mathcal{O}(h)$$

$E = \frac{1}{g^2}\langle I|\mathcal{V}(i)|\Psi\rangle$ from the soft dilaton theorem

$$
\begin{aligned}
&-\frac{1}{g^2}\left[\frac{1}{2}\langle\Psi|Q|\Psi\rangle + \frac{1}{3}\langle\Psi|\Psi*\Psi\rangle + h\langle I|\mathcal{V}(i)|\Psi\rangle\right] \\
&= -\frac{1+h}{g^2}\left[\frac{1}{2}\langle\Psi''|Q|\Psi''\rangle + \frac{1}{3}\langle\Psi''|\Psi''*\Psi''\rangle\right] + \mathcal{O}(h^2) ,
\end{aligned}
$$

Substituting a classical solution $|\Psi_{\rm cl}\rangle$ into it

$$
-E - \frac{h}{g^2}\langle I|\mathcal{V}(i)|\Psi_{\rm cl}\rangle = -(1+h)E + \mathcal{O}(h^2)
$$

and comparing the $\mathcal{O}(h)$ terms

$$
E = \frac{1}{g^2}\langle I|\mathcal{V}(i)|\Psi_{\rm cl}\rangle
$$

Soft dilaton theorem

The soft dilaton theorem is proved in two steps. There exists χ such that

$$\mathcal{V}(i) = \{Q, \chi\}$$

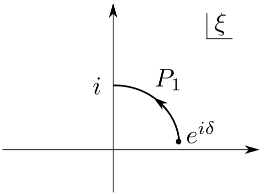
Using this fact, we obtain

$$\begin{aligned} S_h[|\Psi\rangle] &= -\frac{1}{g^2}\left[\frac{1}{2}\langle\Psi|Q|\Psi\rangle + \frac{1}{3}\langle\Psi|\Psi*\Psi\rangle + h\langle I|\mathcal{V}(i)|\Psi\rangle\right] \\ &= -\frac{1}{g^2}\left[\frac{1}{2}\left\langle\Psi'|Q'|\Psi'\right\rangle + \frac{1}{3}\left\langle\Psi'|\Psi'*\Psi'\right\rangle\right] + \mathcal{O}\left(h^2\right) \end{aligned}$$

$$\begin{aligned} \left|\Psi'\right\rangle &\equiv |\Psi\rangle + h\chi|I\rangle \\ Q' &\equiv Q - h\left(\chi - \chi^\dagger\right) \end{aligned}$$

$\mathcal{V}(i) = \{Q, \chi\}$

$$
\begin{aligned}
\chi &\equiv \lim_{\delta \to 0} \left[\int_{P_1} \frac{d\xi}{2\pi i} j(\xi, \bar{\xi}) - \int_{\bar{P}_1} \frac{d\bar{\xi}}{2\pi i} \bar{j}(\xi, \bar{\xi}) - \kappa \left(e^{i\delta}, e^{-i\delta} \right) \right], \\
j(\xi, \bar{\xi}) &\equiv 4 \partial X^0(\xi) \bar{c} \bar{\partial} X^0(\bar{\xi}), \\
\bar{j}(\xi, \bar{\xi}) &\equiv 4 \bar{\partial} X^0(\bar{\xi}) c \partial X^0(\xi), \\
\kappa(\xi, \bar{\xi}) &\equiv \frac{1}{\pi i} \left(X^0(\xi, \bar{\xi}) - X^0(i, -i) \right) \left(c \partial X^0(\xi) - \bar{c} \bar{\partial} X^0(\bar{\xi}) \right).
\end{aligned}
$$



Soft dilaton theorem

There exists $\mathcal{G}$ such that

$$[Q, \mathcal{G}] = \chi - \chi^\dagger$$
$$\langle \mathcal{G}\Psi_1|\Psi_2\rangle + \langle \Psi_1|\mathcal{G}\Psi_2\rangle = \langle \Psi_1|\Psi_2\rangle$$
$$\langle \mathcal{G}\Psi_1|\Psi_2 * \Psi_3\rangle + \langle \Psi_1|\mathcal{G}\Psi_2 * \Psi_3\rangle + \langle \Psi_1|\Psi_2 * \mathcal{G}\Psi_3\rangle = \langle \Psi_1|\Psi_2 * \Psi_3\rangle$$

Using $\mathcal{G}$, we eventually obtain

$$\begin{aligned} S_h[|\Psi\rangle] &\sim -\frac{1}{g^2}\left[\frac{1}{2}\langle \Psi'|\left(Q - h\left(\chi - \chi^\dagger\right)\right)|\Psi'\rangle + \frac{1}{3}\langle \Psi'|\Psi' * \Psi'\rangle\right] \\ &\sim -\frac{1+h}{g^2}\left[\frac{1}{2}\langle \Psi''|Q|\Psi''\rangle + \frac{1}{3}\langle \Psi''|\Psi'' * \Psi''\rangle\right] \end{aligned}$$

$$|\Psi''\rangle \equiv (1 - h\mathcal{G})\,|\Psi'\rangle$$

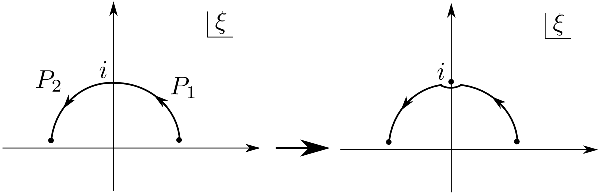
Myers-Penati-Pernici-Strominger

$[Q, \mathcal{G}] = \chi - \chi^\dagger$

$$\mathcal{G} \equiv \lim_{\delta \to 0} \left[\int_{P_1+P_2} \frac{d\xi}{2\pi i} g_\xi \left(\xi, \bar{\xi}\right) - \int_{\bar{P}_1+\bar{P}_2} \frac{d\bar{\xi}}{2\pi i} g_{\bar{\xi}} \left(\xi, \bar{\xi}\right) \right]$$

$$g_\xi \left(\xi, \bar{\xi}\right) \equiv 2 \left(X^0 \left(\xi, \bar{\xi}\right) - X^0 (i, -i) \right) \partial X^0 (\xi)$$

$$g_{\bar{\xi}} \left(\xi, \bar{\xi}\right) \equiv 2 \left(X^0 \left(\xi, \bar{\xi}\right) - X^0 (i, -i) \right) \bar{\partial} X^0 \left(\bar{\xi}\right)$$



Remarks

- The proof is valid for $\mathcal{O}_\Psi$ which does not affect the definition and the manipulations of $\chi, \mathcal{G}$.
- One can obtain the same relation for

$$\mathcal{V} = \frac{1}{\pi i} c\bar{c}\partial X^\mu \bar{\partial} X^\nu h_{\mu\nu}$$

  with $h^\mu_\mu = -1$.

- For actual applications, it is desirable to find a way to derive $E = \frac{1}{g^2} \langle I | \mathcal{V}(i) | \Psi_{\rm cl} \rangle$ more directly using the properties of $\mathcal{G}, \chi$ and the eom.

A more direct proof of $E = \frac{1}{g^2} \langle I | \mathcal{V}(i) | \Psi \rangle$

$\mathcal{G}$ satisfies the following identities:

$$\begin{aligned} \langle \mathcal{G}\Psi | \Psi * \Psi \rangle &= \frac{1}{3} \left[\langle \mathcal{G}\Psi | \Psi * \Psi \rangle + \langle \Psi | \mathcal{G}\Psi * \Psi \rangle + \langle \Psi | \Psi * \mathcal{G}\Psi \rangle \right] \\ &= \frac{1}{3} \langle \Psi | \Psi * \Psi \rangle , \\ \langle \mathcal{G}\Psi | Q | \Psi \rangle &= \frac{1}{2} \langle \Psi | Q | \Psi \rangle + \frac{1}{2} \langle \Psi | [Q, \mathcal{G}] | \Psi \rangle . \end{aligned}$$

From these, we get

$$\begin{aligned} E &= \frac{1}{g^2} \left[\frac{1}{2} \langle \Psi | Q | \Psi \rangle + \frac{1}{3} \langle \Psi | \Psi * \Psi \rangle \right] \\ &= \frac{1}{g^2} \left[\langle \mathcal{G}\Psi | \left(Q | \Psi \rangle + | \Psi * \Psi \rangle \right) - \frac{1}{2} \langle \Psi | [Q, \mathcal{G}] | \Psi \rangle \right] \end{aligned}$$

A more direct proof

$$E = \frac{1}{g^2}\left[\langle\mathcal{G}\Psi|\left(Q|\Psi\rangle + |\Psi*\Psi\rangle\right) - \frac{1}{2}\langle\Psi|\left[Q,\mathcal{G}\right]|\Psi\rangle\right]$$

eom implies

$$\begin{aligned} E &= -\frac{1}{2g^2}\langle\Psi|\left[Q,\mathcal{G}\right]|\Psi\rangle \\ &= -\frac{1}{2g^2}\langle\Psi|\left(\chi-\chi^\dagger\right)|\Psi\rangle \\ &= -\frac{1}{g^2}\langle I|\chi|\Psi*\Psi\rangle \\ &= \frac{1}{g^2}\langle I|\chi Q|\Psi\rangle \\ &= \frac{1}{g^2}\langle I|\mathcal{V}(i)|\Psi\rangle \end{aligned}$$

Remarks

- If eom is not satisfied

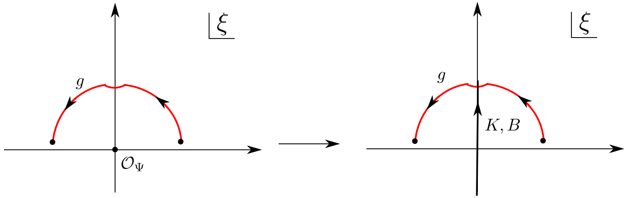
$$Q|\Psi\rangle + |\Psi * \Psi\rangle = |\Gamma\rangle \neq 0\,,$$

$$\begin{aligned} E &= \frac{1}{g^2}\left[\langle \mathcal{G}\Psi|\left(Q|\Psi\rangle + |\Psi * \Psi\rangle\right) - \frac{1}{2}\langle\Psi|[Q,\mathcal{G}]|\Psi\rangle\right] \\ &= -\frac{1}{2g^2}\langle\Psi|[Q,\mathcal{G}]|\Psi\rangle + \frac{1}{g^2}\langle\mathcal{G}\Psi|\Gamma\rangle \\ &= -\frac{1}{g^2}\langle I|\chi|\Psi * \Psi\rangle + \frac{1}{g^2}\langle\mathcal{G}\Psi|\Gamma\rangle \\ &= \frac{1}{g^2}\langle I|\mathcal{V}(i)|\Psi\rangle - \frac{1}{g^2}\langle I|\chi|\Gamma\rangle + \frac{1}{g^2}\langle\mathcal{G}\Psi|\Gamma\rangle \end{aligned}$$

§3 Solutions with K, B

Most of the solutions since Schnabl's one involve

$$K \equiv \int \frac{d\xi}{2\pi i} \frac{\pi}{2} \left(1+\xi^2\right) T(\xi)$$
$$B \equiv \int \frac{d\xi}{2\pi i} \frac{\pi}{2} \left(1+\xi^2\right) b(\xi)$$



The definitions and the manipulations of operators $\mathcal{G}, \chi$ are affected by the presence of K, B.

Okawa type solutions

As an example of such solutions, let us consider Okawa type solutions (Okawa, Erler).

$$\Psi = F(K)\, c \frac{KB}{1 - F(K)^2} c F(K)$$

$$\begin{aligned} K &\equiv \int \frac{d\xi}{2\pi i} \frac{\pi}{2} \left(1 + \xi^2\right) T(\xi) \left|I\right\rangle \\ B &\equiv \int \frac{d\xi}{2\pi i} \frac{\pi}{2} \left(1 + \xi^2\right) b(\xi) \left|I\right\rangle \\ c &\equiv \frac{1}{\pi} c(1) \left|I\right\rangle \end{aligned}$$

Okawa type solutions

As an example of such solutions, let us consider Okawa type solutions (Okawa, Erler).

$$\Psi = F(K)\, c \frac{KB}{1 - F(K)^2} c F(K)$$

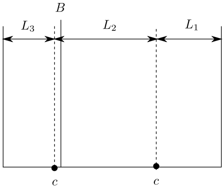
$$F(K) = \int_0^\infty dL e^{-LK} f(L)$$

$$\frac{K}{1 - F^2} = \int_0^\infty dL e^{-LK} \tilde{f}(L)$$

$$\Psi = \int dL_1 dL_2 dL_3 e^{-L_1 K} c e^{-L_2 K} B c e^{-L_3 K} f(L_1) \tilde{f}(L_2) f(L_3)$$

wedge state with insertions

$$\Psi = \int dL_1 dL_2 dL_3 e^{-L_1 K} c e^{-L_2 K} B c e^{-L_3 K} f(L_1) \tilde{f}(L_2) f(L_3)$$

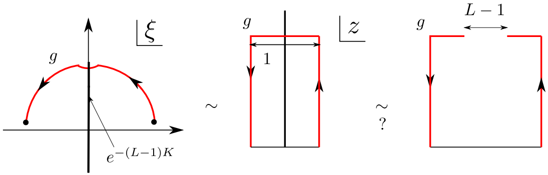


$$\Psi = \int_0^\infty dL e^{-LK} \psi(L) = \int_0^\infty dL e^{-LK} \mathcal{L}^{-1}\{\Psi\}(L)$$

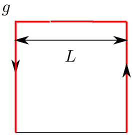
$$\psi(L) \equiv \int dL_1 dL_2 dL_3 \delta(L - L_1 - L_2 - L_3) \times c(L_2 + L_3) Bc(L_3) f(L_1) \tilde{f}(L_2) f(L_3)$$

definition of $\mathcal{G}$

How should we define $\mathcal{G}$ acting on wedge states?
One way to do is



We rather take



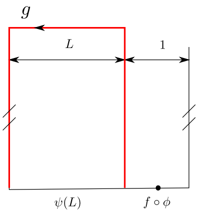
definition of $\mathcal{G}$

$\mathcal{G}\Psi$ is defined so that for test state $|\phi\rangle \equiv \phi(0)|0\rangle$

$$\langle\phi|\mathcal{G}\Psi\rangle \equiv \int_0^\infty dL \left\langle e^{LK} f\circ\phi(0)\, e^{-LK}\mathcal{G}(L)\psi(L)\right\rangle_{C_{L+1}}$$

$$\begin{aligned} f(\xi) &\equiv \frac{2}{\pi}\arctan\xi \\ \mathcal{G}(L) &\equiv \int\frac{dz}{2\pi i}g_z(z,\bar{z}) - \int\frac{d\bar{z}}{2\pi i}g_{\bar{z}}(z,\bar{z}) \end{aligned}$$

$\psi(L)$ in the correlation function denotes the operator corresponding to the string field $\psi(L)$.



$E = \frac{1}{g^2} \langle I | \mathcal{V}(i) | \Psi \rangle$

In order to obtain $E = \frac{1}{g^2} \langle I | \mathcal{V}(i) | \Psi \rangle$, we need

$$\langle \mathcal{G}\Psi_1 | \Psi_2 \rangle + \langle \Psi_1 | \mathcal{G}\Psi_2 \rangle = \langle \Psi_1 | \Psi_2 \rangle$$
$$\langle \mathcal{G}\Psi_1 | \Psi_2 * \Psi_3 \rangle + \langle \Psi_1 | \mathcal{G}\Psi_2 * \Psi_3 \rangle + \langle \Psi_1 | \Psi_2 * \mathcal{G}\Psi_3 \rangle = \langle \Psi_1 | \Psi_2 * \Psi_3 \rangle$$
$$[Q, \mathcal{G}] | \Psi \rangle = \left(\chi - \chi^\dagger \right) | \Psi \rangle$$

If all these are satisfied, we get $E = \frac{1}{g^2} \langle I | \mathcal{V}(i) | \Psi \rangle$.

- It is straightforward to show the first two for $\mathcal{G}$ defined in the previous slide.
- Showing the third one is not straightforward.

$[Q, \mathcal{G}] |\Psi\rangle = (\chi - \chi^\dagger) |\Psi\rangle$

$$
\begin{aligned}
[Q, \mathcal{G}] \Psi &= \int dL e^{-LK} [Q, \mathcal{G}(L)] \psi(L) \\
&\quad + \int dL e^{-LK} \mathcal{G}(L) \left\{ Q \mathcal{L}^{-1} \{\Psi\}(L) - \mathcal{L}^{-1} \{Q\psi\}(L) \right\}
\end{aligned}
$$

$$
Q \mathcal{L}^{-1} \{\Psi\}(L) - \mathcal{L}^{-1} \{Q\psi\}(L) = -e^{LK} \partial \left(e^{-LK} \alpha(L) \right) - \delta(L) \alpha(0)
$$

$$
\begin{aligned}
\alpha(L) &\equiv \int dL_1 dL_2 dL_3 \delta(L - L_1 - L_2 - L_3) \\
&\quad \times c(L_2 + L_3) c(L_3) f(L_1) \tilde{f}(L_2) f(L_3)
\end{aligned}
$$

Assuming $\alpha(\infty) = 0$ and $\alpha(0)$ is finite, we are able to get $[Q, \mathcal{G}] |\Psi\rangle = (\chi - \chi^\dagger) |\Psi\rangle$.

Solutions with K, B

In summary, it is possible to show $E = \frac{1}{g^2} \langle I | \mathcal{V}(i) | \Psi \rangle$, even for the solutions with K, B provided $\alpha(\infty) = 0$ and $\alpha(0)$ is finite.

$$\begin{aligned} \alpha(L) &\equiv \int dL_1 dL_2 dL_3 \delta(L - L_1 - L_2 - L_3) \\ &\quad \times c(L_2 + L_3)\, c(L_3)\, f(L_1)\, \tilde{f}(L_2)\, f(L_3) \end{aligned}$$

These conditions for $\alpha(L)$ are concerned with the regularity of the function $F(K)$ for $K \sim 0, K \sim \infty$.

- If $Q|\Psi\rangle + |\Psi * \Psi\rangle = |\Gamma\rangle \neq 0$, we obtain

$$E = \frac{1}{g^2} \langle I | \mathcal{V}(i) | \Psi \rangle - \frac{1}{g^2} \langle I | \chi | \Gamma \rangle + \frac{1}{g^2} \langle \mathcal{G}\Psi | \Gamma \rangle$$

§5 Conclusions

- We have given a proof of

$$E = \frac{1}{g^2} \langle I | \mathcal{V}(i) | \Psi \rangle$$

  for a classical solution Ψ of Witten's SFT.
- The gauge invariant observables seem to have enough information to reproduce energy, boundary state, etc.. Kudrna-Maccaferri-Schnabl
- This identity can be applied to BMT, Murata-Schnabl solutions.
- It is straightforward to generalize the proof to modified cubic SSFT. T. Baba
- Masuda solution? Masuda-Noumi-Takahashi

Thank you

Murata-Schnabl solutions

In order to calculate the gauge invariant observables

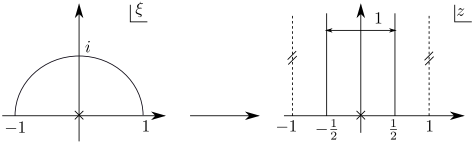
$$\Psi \to \Psi_\epsilon \equiv F^2 (K+\epsilon) \, cB \frac{K+\epsilon}{1 - F^2 (K+\epsilon)} c$$

The energy with this regularization can be calculated by our method:

$$E = \frac{1}{g^2} \left(\frac{N-1}{2\pi^2} - R_N \right)$$

$$R_N \equiv \begin{cases} -\frac{i}{8\pi^3} \sum_{k=0}^{N-2} \frac{N!}{k!(k+2)!(N-2-k)!} \left((2\pi i)^{k+2} - (-2\pi i)^{k+2} \right) & , \ (N \geq 1) \ , \\ \frac{i}{8\pi^3} \sum_{k=0}^{-N-1} \frac{(1-N)!}{k!(k+2)!(-N-1-k)!} \left((2\pi i)^{k+2} - (-2\pi i)^{k+2} \right) & , \ (N \leq 0) \ . \end{cases}$$

Sliver frame, wedge states



$$\xi = \tan \frac{\pi z}{2}$$

$$K = \int \frac{dz}{2\pi i} T(z)$$