Dimensional regularization of light-cone gauge superstring field theory and multiloop amplitudes

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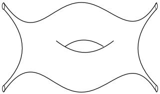
String Field Theory

- SFT should reproduce the scattering amplitudes calculated by the first-quantized formalism.

$$A = \sum_{\text{worldsheet}}$$



- For bosonic strings, there are several SFT's which are shown to reproduce the first quantized results.
  - Light-cone gauge SFT, Witten's cubic SFT, Zwiebach's closed SFT

Supserstring Field Theory

- For superstrings, there are no such SFT's yet.
- This is because the superstring amplitudes calculated by using SFT's usually involves picture changing operators (PCO), but the way to calculate amplitudes using PCO's has not been well-understood until recently.
- Recently, developing the techniques by Saroja-Sen and Sen, Sen and Witten have given a prescription to calculate superstring amplitudes using PCO's. (arXIv:1504.00609)

In this talk

I would like to explain that using light-cone gauge SFT for superstrings (NSR formalism)

- it is possible to calculate multiloop amplitudes
- the results coincide with those obtained by Sen-Witten prescription
  - NS-NS sector external lines, even spin structure

a generalization of the works done with Y. Baba and K. Murakami

JHEP 0910 035, 1001 119, 1008 102, 1101 008, 1107 090, 1309 053

Outline

§1 LC gauge super SFT

§2 Dimensional regularization

§3 $d \to 10$

§4 Conclusions and discussions

§1 Light-cone gauge super SFT

LC gauge SFT for Type II superstrings (NSR) Mandelstam, S.J. Sin

heterotic strings Gross-Periwal

$$S = \int \left[\frac{1}{2}\Phi \cdot \left(i\partial_t - \frac{L_0 + \tilde{L}_0 - 1}{p^+} \right) \Phi + \frac{g}{3}\Phi \cdot (\Phi * \Phi) \right] + (\text{Ramond sectors})$$

- string field: $\Phi\left[t, p^+, X^i(\sigma)\right]$ $(t = x^+)$
- three string vertex GO

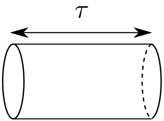
Perturbation theory

Feynmn rule

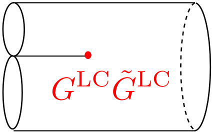
- propagator

$$i\int_0^\infty d\tau e^{-\tau\frac{L_0+\tilde{L}_0-1}{\alpha}}$$

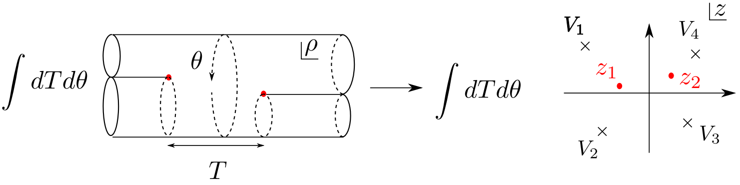
$$\int d\tau$$



- vertex



Feynman diagram



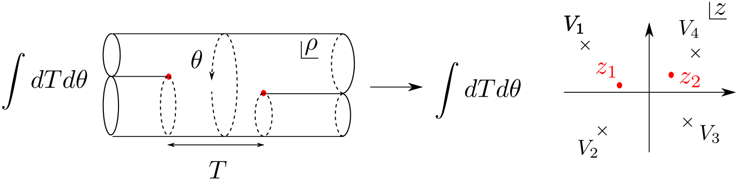
The worldsheet can be mapped to a closed Riemann surface with punctures.

$$\rho(z) = \sum_{r=1}^{4} p_r^{+} \ln(z - Z_r)$$

- V_r is at $z = Z_r$
- z_I $(I = 1, 2)$: interaction points $(\partial\rho(z_I) = 0)$

Feynman diagram

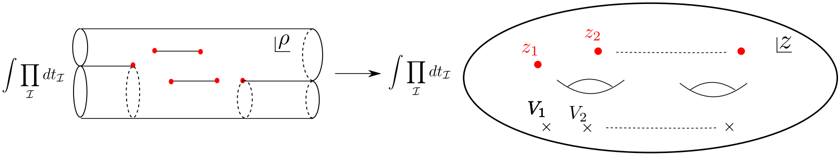
Using the map, the amplitude can be expressed in terms of the vertex operators



- The worldsheet theory is with $c = 12 \neq 0$, and we have an anomaly factor $e^{-\Gamma}$.

$$A^{\mathrm{LC}} = \sum_{\text{Channels}} \int dT d\theta \left\langle \prod_{I=1}^{2} \left| \left(\partial^2 \rho\right)^{-\frac{3}{4}} G^{\mathrm{LC}}\left(z_I\right) \right|^2 \prod_{r=1}^{4} V_r^{\mathrm{LC}} \right\rangle^{X^i, \psi^i} e^{-\Gamma}$$

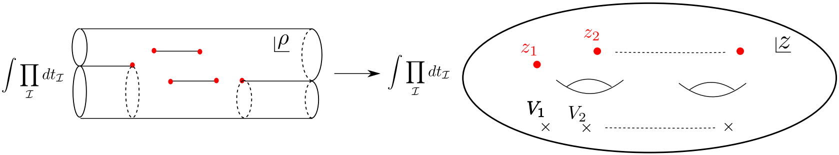
General amplitudes



$$\rho(z)=\sum_{r=1}^{N} p_r^{+}\left[\ln E(z,Z_r)-2\pi i\int_{P_0}^{z}\omega\frac{1}{\mathrm{Im}\Omega}\mathrm{Im}\int_{P_0}^{Z_r}\omega\right]$$

$E(z,w)$	:	prime form $\sim z-w$
ω	:	Abelian differential
Ω	:	period matrix

General amplitudes

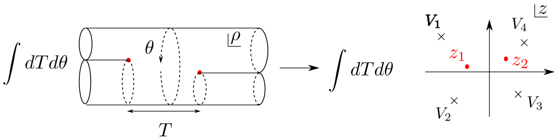


$$A^{\mathrm{LC}} = \sum_{\mathrm{Channels}} \int \prod_{\mathcal{I}} dt_{\mathcal{I}} \left\langle \prod_{I=1}^{2g-2+N} \left| (\partial^2 \rho)^{-\frac{3}{4}} G^{\mathrm{LC}}(z_I) \right|^2 \prod_{r=1}^{N} V_r^{\mathrm{LC}} \right\rangle^{X^i, \psi^i} e^{-\Gamma}$$

$$e^{-\Gamma} \propto \prod_{r=1}^{N} \left[\alpha_r^{-1} (g^{\mathrm{A}}_{Z_r \bar{Z}_r})^{-\frac{1}{2}} e^{-\mathrm{Re} \bar{N}^{rr}_{00}} \right] \prod_{I=1}^{2g-2+N} \left[(g^{\mathrm{A}}_{z_I \bar{z}_I})^{-\frac{1}{2}} \left| \partial^2 \rho(z_I) \right|^{-\frac{1}{2}} \right]$$

GO

Contact term divergence



$$A^{\mathrm{LC}} = \sum_{\text{Channels}} \int dTd\theta \left\langle \prod_{I=1}^{2} \left| \left(\partial^2\rho\right)^{-\frac{3}{4}} G^{\mathrm{LC}}(z_I) \right|^2 \prod_{r=1}^{4} V_r^{\mathrm{LC}} \right\rangle^{X^i,\psi^i} e^{-\Gamma}$$

- When z_1 and z_2 come closed to each other the integral diverges. GO
- General ampllitudes also diverge.

We cannot calculate even the tree level amplitudes.

Remarks

- Witten's cubic SFT for superstrings also suffers from this problem, but there are several ways to avoid this problem. (modified cubic, Berkovits, Erler-Konopka-Sachs, ...)
- In the case of multiloop amplitudes, this divergence corresponds to the so-called spurious singularities in the superstring perturbation theory using PCO's. GO

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  In the case of LC gauge amplitudes, all we need to deal with are the contact terms.

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  In the case of LC gauge amplitudes, all we need to deal with are the contact terms.

  We will do so by dimensional regularization.

§2 Dimensional regularization

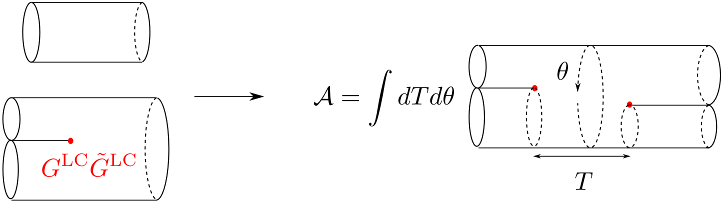
Light-cone gauge SFT can be formulated in any d GO

$$S = \int \left[\frac{1}{2}\Phi \cdot \left(i\partial_t - \frac{L_0 + \tilde{L}_0 - \frac{d-2}{8}}{p^+} \right) \Phi + \frac{g}{3}\Phi \cdot (\Phi * \Phi) \right] + (\text{Ramond sectors})$$

- LC gauge SFT is a completely gauge fixed theory.
- The Lorentz invariance is broken.

Baba-N.I.-Murakami

Amplitudes

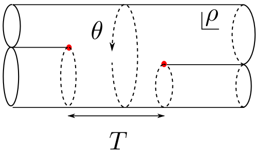


$$A_d^{\mathrm{LC}} = \sum_{\mathrm{Channels}} \int \prod_{\mathcal{I}} dt_{\mathcal{I}} \left\langle \prod_{I=1}^{2g-2+N} \left| \left(\partial^2 \rho\right)^{-\frac{3}{4}} G^{\mathrm{LC}}\left(z_I\right) \right|^2 \prod_{r=1}^{N} V_r^{\mathrm{LC}} \right\rangle^{X^i, \psi^i} e^{-\frac{d-2}{8}\Gamma}$$

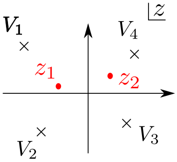
- The amplitude is modular invariant.

Contact term

$\int dT d\theta$



$\longrightarrow \int dT d\theta$



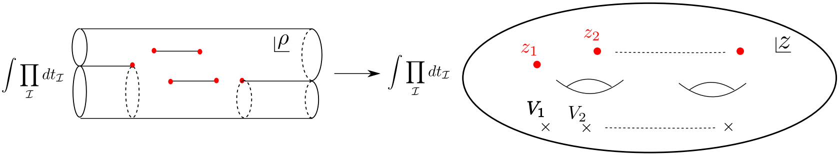
$$A_d^{\mathrm{LC}} = \sum_{\mathrm{Channels}} \int dT d\theta \left\langle \prod_{I=1}^{2} \left| \left(\partial^2 \rho\right)^{-\frac{3}{4}} G^{\mathrm{LC}}\left(z_I\right) \right|^2 \prod_{r=1}^{4} V_r^{\mathrm{LC}} \right\rangle^{X^i, \psi^i} e^{-\frac{d-2}{8}\Gamma}$$

- For $z_1 \sim z_2$

$$A_d^{\mathrm{LC}} \sim \int d^2\left(z_1 - z_2\right) \left|z_1 - z_2\right|^{-\frac{d-2}{8}-5}$$

By taking $-d$ to be large enough, the integral is convergent.

Contact term



$$A_d^{\mathrm{LC}} = \sum_{\mathrm{Channels}} \int \prod_{\mathcal{I}} dt_{\mathcal{I}} \left\langle \prod_{I=1}^{2g-2+N} \left| \left(\partial^2 \rho\right)^{-\frac{3}{4}} G^{\mathrm{LC}}\left(z_I\right) \right|^2 \prod_{r=1}^{N} V_r^{\mathrm{LC}} \right\rangle^{X^i, \psi^i} e^{-\frac{d-2}{8}\Gamma}$$

- When n of z_I come close to each other

$$A_d^{\mathrm{LC}} \sim \int d^2 z \, |z|^{-\frac{d-2}{16} n(n-1) + \frac{1}{2} n^2 - \frac{3}{2} n - 4}$$

We can regularize the divergence by taking $-d$ large enough.

Simple dimensional regularization does not work

$$A_d^{\mathrm{LC}} = \sum_{\text{Channels}} \int \prod_{\mathcal{I}} dt_{\mathcal{I}} \left\langle \prod_{I=1}^{2g-2+N} \left| \left(\partial^2 \rho\right)^{-\frac{3}{4}} G^{\mathrm{LC}}\left(z_I\right) \right|^2 \prod_{r=1}^{N} V_r^{\mathrm{LC}} \right\rangle^{X^i, \psi^i} e^{-\frac{d-2}{8}\Gamma}$$

- The integrand includes the fermion partition function $\left(Z^{\psi}\right)^{d-2} \propto \left(\vartheta\left[\alpha\right](0)\right)^{d-2}$.
- Taking $-d$ large, the integrand has a singularity similar to the spurious singularity of the second type.

One can avoid this problem

- What actually matters is the Virasoro central charge $\hat{c}$ rather than the number of the spacetime coordinates

$$A = \int \prod_{\mathcal{I}} dt_{\mathcal{I}} \left\langle \prod_{I=1}^{2g-2+N} \left| \left(\partial^2 \rho\right)^{-\frac{3}{4}} G^{\mathrm{LC}}(z_I) \right|^2 \prod_{r=1}^{N} V_r^{\mathrm{LC}} \right\rangle^{X^i,\psi^i} e^{-\frac{\hat{c}-2}{16}\Gamma}$$

- By taking the worldsheet theory to be for example

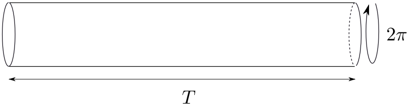
$$\begin{array}{ccccccc} & X^i, \psi^i & & SU(2)\text{super WZW} \times 2M & & \left(\hat{b}, \hat{c}, \hat{\beta}, \hat{\gamma}\right) \times 3M \\ c = & 12 & + & \left(\frac{3k}{k+2} + \frac{3}{2}\right) \times 2M & + & (-3) \times 3M \end{array}$$

where $\left(\hat{b}, \hat{c}, \hat{\beta}, \hat{\gamma}\right)$ are of weight $\left(1, 0, \frac{1}{2}, \frac{1}{2}\right)$, the fermion partition function is always $\left(Z^{\psi}\right)^8$ instead of $\left(Z^{\psi}\right)^{d-2}$.

Remarks

- The dimensional regularization works also as a IR regularization for strig theory

$$S = \int \left[\frac{1}{2} \Phi \cdot \left(i\partial_\tau - \frac{L_0 + \tilde{L}_0 - \frac{\hat{c}-2}{8}}{p^+} \right) \Phi + \frac{g}{3} \Phi \cdot (\Phi * \Phi) \right]$$



$$\sim e^{\frac{d-2}{8} T}$$

§3 $d \to 10$

- The contact term divergences are regularized by considering the theory in d dimensions.
- We can define the amplitudes for $-d$ large enough and analytically continue d to 10.
- We would like to know what happens in the limit $d \to 10$.
  - If the limit diverges, we need to add counterterms.

$d \to 10$

With the help of the work by Sen and Witten, we can show that the limit $d \to 10$ is smooth, assuming

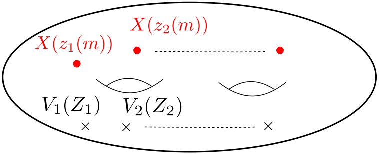
- the superstring perturbation theory is without the IR divergence coming from the boundary of the moduli space
- NS-NS external lines, even spin structure (parity nonviolating amplitudes)

Dimensional regularization

Even for $d \neq 10$, following the same procedure as that in the critical case

$$\begin{aligned} A_d^{\mathrm{LC}} &= \int \prod_{\mathcal{I}} dt_{\mathcal{I}} \left\langle \prod_{I=1}^{2g-2+N} \left| \left(\partial^2 \rho\right)^{-\frac{3}{4}} G^{\mathrm{LC}}\left(z_I\right) \right|^2 \prod_{r=1}^{N} V_r^{\mathrm{LC}} \right\rangle^{X^i,\psi^i} e^{-\frac{d-2}{16}\Gamma} \\ &= \int \prod_K dm_K \left\langle \prod_K \oint \left(\mu_K b + \bar{\mu}_K \bar{b}\right) \prod_{I=1}^{2g-2+N} X(z_I)\bar{X}(\bar{z}_I) \prod_{r=1}^{N} V_r^{\mathrm{conf.}} \right\rangle^{X^\mu,\psi^\mu,\mathrm{ghosts}} \end{aligned}$$

$$\int \prod_K dm_K$$



but with a nontrivial CFT for $X^{\pm}, \psi^{\pm}$ ($X^{\pm}$ CFT).

$X^{\pm}$ CFT

$$S_{X^{\pm}} = -\frac{1}{2\pi}\int d^2 z d\theta d\bar{\theta}\left(\bar{D}X^{+}DX^{-} + \bar{D}X^{-}DX^{+}\right) + \frac{d-10}{8}\Gamma_{\mathrm{super}}[\Phi]$$

$$X^{\pm} \equiv x^{\pm} + i\theta\psi^{\pm} + i\bar{\theta}\tilde{\psi}^{\pm} + i\theta\bar{\theta}F^{\pm}$$

$$\Gamma_{\mathrm{super}}[\Phi] = -\frac{1}{2\pi}\int d^2 z d\theta d\bar{\theta}\left(\bar{D}\Phi D\Phi + \theta\bar{\theta}\hat{g}_{z\bar{z}}\hat{R}\Phi\right)$$

$$\Phi \equiv \ln\left|\partial X^{+} - \frac{\partial D X^{+} D X^{+}}{(\partial X^{+})^2}\right|^2 - \ln \hat{g}_{z\bar{z}}$$

- This theory can be formulated in the case $\langle \partial_m X^{+} \rangle \neq 0$.
- In the case of the LC gauge amplitudes, we always have $\prod e^{-ip_r^{+}X^{-}}$ $(p_r^{+} \neq 0)$ and $\langle \partial_m X^{+} \rangle \neq 0$.

$X^{\pm}$ CFT

$$
\begin{aligned}
S_{X^{\pm}} &= -\frac{1}{2\pi}\int d^2z d\theta d\bar{\theta}\left(\bar{D}X^{+}DX^{-}+\bar{D}X^{-}DX^{+}\right)+\frac{d-10}{8}\Gamma_{\text{super}}\left[\Phi\right] \\
T\left(z,\theta\right) &= G\left(z\right)+\theta T\left(z\right) \\
&= \frac{1}{2}:\partial X^{+}DX^{-}\left(\mathbf{z}\right):+\frac{1}{2}:DX^{+}\partial X^{-}\left(\mathbf{z}\right):-\frac{d-10}{4}S\left(\mathbf{z},\boldsymbol{X}^{+}\right)
\end{aligned}
$$

- It is a superconformal field theory with $\hat{c}=12-d$. GO
- The worldsheet theory becomes BRST invariant

		$X^{\pm}$		X^i		ghosts		
$\hat{c}$	$=$	$12-d$	$+$	$d-2$	$-$	10	$=$	0

BRST invariant worldsheet theory

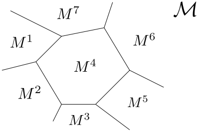
$$
\begin{aligned}
A_d^{\mathrm{LC}} &= \int_{\mathcal{M}} \prod_K dm_K \left\langle \prod_K \oint (\mu_K b + \bar{\mu}_K \bar{b}) \prod_{I=1}^{2g-2+N} X(z_I)\bar{X}(\bar{z}_I) \prod_{r=1}^{N} V_r^{\mathrm{conf.}} \right\rangle^{X^\mu,\psi^\mu,\mathrm{ghosts}} \\
&= \int_{\mathcal{M}} \omega_d^{\mathrm{LC}}
\end{aligned}
$$

- This expression is well-defined with $-d$ large enough.
- With the BRST invariant worldsheet theory, it is also possible to calculate amplitudes following the prescription by Sen and Witten.

With a BRST invariant worldsheet theory

One can define amplitudes using PCO. (Sen, Sen-Witten)

- Divide the moduli space $\mathcal{M}$ into polyhedra M^α.



- In each M^α, put the PCO's avoiding spurious singularities.
- The amplitudes are given in the form

$$A^{\text{SW}} = \sum_\alpha \int_{M^\alpha} \omega^\alpha + A^{\text{vertical}}$$

where A^{vertical} is given as a sum of integrals over ∂M^α and their submanifolds.

$A_d^{\rm SW}$ for dimensionally regularized amplitudes

$$\begin{aligned} A_d^{\rm LC} &= \int_{\mathcal{M}} \prod_K dm_K \left\langle \prod_K \oint (\mu_K b + \bar{\mu}_K \bar{b}) \prod_{I=1}^{2g-2+N} X(z_I)\bar{X}(\bar{z}_I) \prod_{r=1}^{N} V_r^{\rm conf.} \right\rangle^{X^\mu,\psi^\mu,\rm ghosts} \\ &= \int_{\mathcal{M}} \omega_d^{\rm LC} \end{aligned}$$

By putting PCO's at $z = z_I$ except for those polyhedra $(\tilde{M}^\alpha)$ which include the spurious singularities,

$$A_d^{\rm SW} = \int_{\mathcal{M}-\sum_\alpha \tilde{M}^\alpha} \omega_d^{\rm LC} + \sum_\alpha \int_{\tilde{M}^\alpha} \tilde{\omega}^\alpha + A^{\rm vertical}$$

- $A_d^{\rm SW}$ does not depend on the way to choose $\tilde{M}^\alpha$.
- $A_d^{\rm SW}$ is analytic in d and the limit $d \to 10$ is smooth.

$A_d^{\mathrm{SW}} = A_d^{\mathrm{LC}}$

$$A_d^{\mathrm{SW}} = \int_{\mathcal{M}-\sum_\alpha \tilde{M}^\alpha} \omega_d^{\mathrm{LC}} + \sum_\alpha \int_{\tilde{M}^\alpha} \tilde{\omega}^\alpha + A^{\mathrm{vertical}}$$

For $-d$ sufficiently large, taking the limit where the sizes of $\tilde{M}^\alpha$ go to zero

$$\begin{cases} \int_{\tilde{M}^\alpha} \tilde{\omega}^\alpha & \to 0 \\ A^{\mathrm{vertical}} & \to 0 \\ \int_{\mathcal{M}-\sum_\alpha \tilde{M}^\alpha} \omega_d^{\mathrm{LC}} & \to \int_{\mathcal{M}} \omega_d^{\mathrm{LC}} = A_d^{\mathrm{LC}} \end{cases}$$

and

$$A_d^{\mathrm{SW}} = A_d^{\mathrm{LC}}$$

$d \to 10$

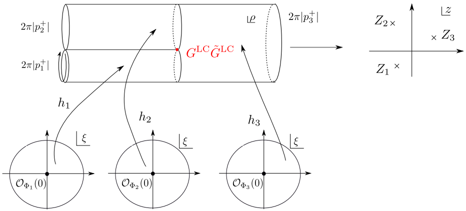
$$A_d^{\mathrm{SW}} = A_d^{\mathrm{LC}}$$

for sufficiently large $-d$ and the limit $d \to 10$ for A_d^{SW} is convergent. Therefore the limit $d \to 10$ for the dimensionally regularized amplitudes are finite and coincides with the amplitude A_{10}^{SW}.

§5 Conclusions and discussions

- Dimensional regularization of the light-cone gauge super SFT can be used to reproduce the results of the first quantized formalism.
- Dimensional regularization in Witten's cubic SFT?
- Nonperturbative calculations by SFT?

Three-string vertex ▸ BACK



The total central charge is 12 and we have to include the anomaly factor

$$\int \Phi_1 \cdot (\Phi_2 * \Phi_3) = \int dt \prod_{r=1}^{3} \left(\frac{p_r^+ dp_r^+}{4\pi} \right) \delta\left(\sum_{r=1}^{3} p_r^+ \right) \left(p_1^+ p_2^+ p_3^+ \right)^{-\frac{1}{2}} e^{-\sum_r \frac{1}{p_r^+} \sum_{s=1}^{3} p_s^+ \ln\left| p_s^+ \right|}$$

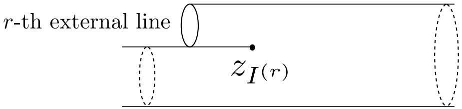
$$\times \left\langle \left| \partial^2 \rho\left(z_I \right) \right|^{-\frac{3}{2}} G^{\rm LC}\left(z_I \right) \bar{G}^{\rm LC}\left(\bar{z}_I \right) \right.$$

$$\left. \times\ \rho^{-1} h_1 \circ \mathcal{O}_{\Phi_1(t,\alpha_1)} \rho^{-1} h_2 \circ \mathcal{O}_{\Phi_2(t,\alpha_2)} \rho^{-1} h_3 \circ \mathcal{O}_{\Phi_3(t,\alpha_3)} \right\rangle_{\mathbb{C}}$$

Anomaly factor ▸ BACK

$$e^{-\Gamma} \propto \prod_{r=1}^{N} \left[\alpha_r^{-1} (g^{\mathrm{A}}_{Z_r \bar{Z}_r})^{-\frac{1}{2}} e^{-\mathrm{Re} \bar{N}^{rr}_{00}} \right] \prod_{I=1}^{2g-2+N} \left[(g^{\mathrm{A}}_{z_I \bar{z}_I})^{-\frac{1}{2}} \left| \partial^2 \rho(z_I) \right|^{-\frac{1}{2}} \right]$$

- $r = 1, \ldots, N$ label the punctures
- $I = 1, \ldots, 2g - 2 + N$ label the interaction points, where $\partial \rho(z_I) = 0$.
- $g^{\mathrm{A}}_{z\bar{z}}$: Arakelov metric on the surface
- $\bar{N}^{rr}_{00} \equiv \frac{1}{p_r^+} \left(\rho(z_{I(r)}) - \lim_{z \to Z_r} \left(\rho(z) - p_r^+ \ln(z - Z_r) \right) \right)$



Contact term ▸ BACK

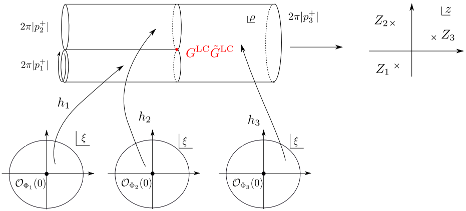
$$A^{\mathrm{LC}} = \sum_{\text{Channels}} \int dT d\theta \left\langle \prod_{I=1}^{2} \left| \left(\partial^2 \rho\right)^{-\frac{3}{4}} G^{\mathrm{LC}}\left(z_I\right) \right|^2 \prod_{r=1}^{4} V_r^{\mathrm{LC}} \right\rangle^{X^i, \psi^i} e^{-\Gamma}$$

For $z_1 \sim z_2$

$$\begin{aligned} G^{\mathrm{LC}}(z_1) G^{\mathrm{LC}}(z_2) &\sim (z_1 - z_2)^{-3} \\ \partial^2 \rho(z_I) &\sim z_1 - z_2 \\ e^{-\Gamma} &\sim |z_1 - z_2|^{-1} \\ dT d\theta &\sim d^2 (z_1 - z_2) |z_1 - z_2|^4 \end{aligned}$$

$$A^{\mathrm{LC}} \sim \int d^2 (z_1 - z_2) |z_1 - z_2|^{-6}$$

Three-string vertex ▸ BACK



The total central charge is $\frac{3}{2}(d-2)$ and the anomaly factor becomes

$$\int \Phi_1 \cdot (\Phi_2 * \Phi_3) = \int dt \prod_{r=1}^{3} \left(\frac{p_r^+ dp_r^+}{4\pi} \right) \delta \left(\sum_{r=1}^{3} p_r^+ \right)$$

$$\times \left(p_1^+ p_2^+ p_3^+ \right)^{-\frac{d-2}{16}} e^{-\frac{d-2}{8} \sum_r \frac{1}{p_r^+} \sum_{s=1}^{3} p_s^+ \ln|p_s^+|}$$

$$\times \left\langle \left| \partial^2 \rho(z_I) \right|^{-\frac{3}{2}} G^{LC}(z_I) \bar{G}^{LC}(\bar{z}_I) \right.$$

$$\left. \times \; \rho^{-1} h_1 \circ \mathcal{O}_{\Phi_1(t,\alpha_1)} \rho^{-1} h_2 \circ \mathcal{O}_{\Phi_2(t,\alpha_2)} \rho^{-1} h_3 \circ \mathcal{O}_{\Phi_3(t,\alpha_3)} \right\rangle_{\mathbb{C}}$$

Conformal gauge expression

The light-cone gauge amplitudes can be recast into the conformal gauge one.

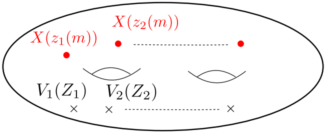
$$
\begin{aligned}
A^{\mathrm{LC}} &= \sum_{\text{Channels}} \int \prod_{\mathcal{I}} dt_{\mathcal{I}} \left\langle \prod_{I=1}^{2g-2+N} \left| \left(\partial^2\rho\right)^{-\frac{3}{4}} G^{\mathrm{LC}}\left(z_I\right) \right|^2 \prod_{r=1}^{N} V_r^{\mathrm{LC}} \right\rangle^{X^i,\psi^i} e^{-\Gamma} \\
&= \int_{\mathcal{M}} \prod_K dm_K \left\langle \prod_K \oint \left(\mu_K b + \bar{\mu}_K \bar{b}\right) \prod_{I=1}^{2g-2+N} X(z_I)\bar{X}(\bar{z}_I) \prod_{r=1}^{N} V_r^{\mathrm{conf.}} \right\rangle^{X^\mu,\psi^\mu,\mathrm{ghosts}}
\end{aligned}
$$

- $X(z) = -e^{\phi}G + c\partial\xi + \frac{1}{4}\partial b\eta e^{2\phi} + \frac{1}{4}\left(2\partial\eta e^{2\phi} + \eta\partial e^{2\phi}\right)$: picture changing operator (PCO) GO
- $\mathcal{M}$: moduli space of the Riemann surface

Conformal gauge expression

The amplitudes from first-quantized formalism has the form

$$\int \prod_K dm_K$$



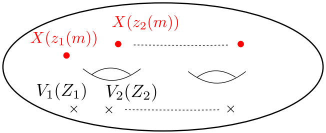
The light-cone gauge amplitude corresponds to a specific way to put the PCO's on the worldsheet.

$$A^{\mathrm{LC}} = \int_{\mathcal{M}} \prod_K dm_K \left\langle \prod_K \oint (\mu_K b + \bar{\mu}_K \bar{b}) \prod_{I=1}^{2g-2+N} X(z_I)\bar{X}(\bar{z}_I) \prod_{r=1}^{N} V_r^{\mathrm{conf.}} \right\rangle^{X^\mu, \psi^\mu, \mathrm{ghosts}}$$

Spurious singularities

The way to calculate superstring amplitudes using PCO's suffers from spurious singularities.

$$\int \prod_K dm_K$$



Superghost correlation function (holomorphic part) GO

$$\left\langle \prod_i e^{\phi}(z_i) \prod_r e^{-\phi}(Z_r) \right\rangle$$

$$\propto \frac{1}{\vartheta[\alpha]\left(\sum z_i - \sum Z_r - 2\triangle\right)} \cdot \frac{\prod_{i,r} E(z_i, Z_r)}{\prod_{i>j} E(z_i, z_j) \prod_{r>s} E(Z_r, Z_s)} \cdot \frac{\prod_r \sigma(Z_r)^2}{\prod_i \sigma(z_i)^2}$$

Spurious singularities

$$\left\langle \prod_i e^{\phi}(z_i) \prod_r e^{-\phi}(Z_r) \right\rangle$$
$$\propto \frac{1}{\vartheta[\alpha]\left(\sum z_i - \sum Z_r - 2\triangle\right)} \cdot \frac{\prod_{i,r} E(z_i, Z_r)}{\prod_{i>j} E(z_i, z_j) \prod_{r>s} E(Z_r, Z_s)} \cdot \frac{\prod_r \sigma(Z_r)^2}{\prod_i \sigma(z_i)^2}$$

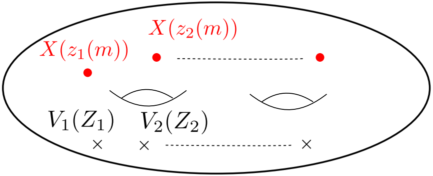
Two kinds of singularities

$$\begin{cases} z_i = z_j \\ \vartheta[\alpha]\left(\sum z_i - \sum Z_r - 2\triangle\right) = 0 \end{cases}$$

At these points, the gauge fixing is not good.

Spurious singularities

$$\int \prod_K dm_K$$



- $z_i = z_j$: PCO's collide
- $\vartheta[\alpha](\sum z_i - \sum Z_r - 2\triangle) = 0$: no interpretation in terms of local operators

The contact term divergence of A^{LC} corresponds to the first type.

A^{LC} involves no singularities of the second type ▸ BACK

$$
\begin{aligned}
A^{\mathrm{LC}} &= \sum_{\text{Channels}} \int \prod_{\mathcal{I}} dt_{\mathcal{I}} \left\langle \prod_{I=1}^{2g-2+N} \left| \left(\partial^2\rho\right)^{-\frac{3}{4}} G^{\mathrm{LC}}\left(z_I\right) \right|^2 \prod_{r=1}^{N} V_r^{\mathrm{LC}} \right\rangle^{X^i,\psi^i} e^{-\Gamma} \\
&= \int \prod_K dm_K \left\langle \prod_K \oint \left(\mu_K b + \bar{\mu}_K \bar{b}\right) \prod_{I=1}^{2g-2+N} X(z_I)\bar{X}(\bar{z}_I) \prod_{r=1}^{N} V_r^{\mathrm{conf.}} \right\rangle^{X^\mu,\psi^\mu,\mathrm{ghosts}}
\end{aligned}
$$

- The first line involves no $\beta\gamma$ system.
- In the second line, $\vartheta\left[\alpha\right](0)$ which comes from the Z^ψ cancels

$$
\frac{1}{\vartheta\left[\alpha\right]\left(\sum z_i - \sum Z_r - 2\triangle\right)} = \frac{1}{\vartheta\left[\alpha\right](0)}
$$

What we have to deal with is only the contact term problem.

$X^{\pm}$ CFT

$$
\begin{aligned}
T(z,\theta) &= G(z) + \theta T(z) \\
&= \frac{1}{2} : \partial X^{+} D X^{-}(\mathbf{z}) : + \frac{1}{2} : D X^{+} \partial X^{-}(\mathbf{z}) : - \frac{d-10}{4} S\left(\mathbf{z}, \boldsymbol{X}^{+}\right)
\end{aligned}
$$

$$
\begin{aligned}
S\left(\mathbf{z}, \boldsymbol{X}^{+}\right) &= \frac{\partial^{2}\Theta^{+}}{D\Theta^{+}} - \frac{2\partial D\Theta^{+}\partial\Theta^{+}}{\left(D\Theta^{+}\right)^{2}} \\
\Theta^{+} &= \frac{DX^{+}}{\left(\partial X^{+}\right)^{\frac{1}{2}}} \\
\mathbf{z} &= (z,\theta) \\
D &= \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial z}
\end{aligned}
$$

$X^{\pm}$ CFT

One can calculate the OPE's:

$$
\begin{aligned}
X^{+}(\mathbf{z})\,X^{+}(\mathbf{z}') &\sim \textbf{regular}\\
X^{+}(\mathbf{z})\,X^{-}(\mathbf{z}') &\sim \ln\left|\mathbf{z}-\mathbf{z}'\right|^{2}\\
X^{-}(\mathbf{z})\,X^{-}(\mathbf{z}') &\sim -\frac{d-10}{4}\Bigg[\frac{\theta-\theta'}{(\mathbf{z}-\mathbf{z}')^{3}}\frac{3DX^{+}}{\left(\partial X^{+}\right)^{3}}(\mathbf{z}')\\
&\quad+\frac{1}{(\mathbf{z}-\mathbf{z}')^{2}}\left(\frac{1}{2\left(\partial X^{+}\right)^{2}}+\frac{4\partial DX^{+}DX^{+}}{\left(\partial X^{+}\right)^{4}}\right)(\mathbf{z}')\\
&\quad+\frac{\theta-\theta'}{(\mathbf{z}-\mathbf{z}')^{2}}\left(-\frac{\partial DX^{+}}{\left(\partial X^{+}\right)^{3}}-\frac{5\partial^{2}X^{+}DX^{+}}{2\left(\partial X^{+}\right)^{4}}\right)(\mathbf{z}')\\
&\quad+\frac{1}{\mathbf{z}-\mathbf{z}'}\left(-\frac{\partial^{2}X^{+}}{2\left(\partial X^{+}\right)^{3}}+\frac{2\partial^{2}DX^{+}DX^{+}}{\left(\partial X^{+}\right)^{4}}-\frac{8\partial^{2}X^{+}\partial DX^{+}DX^{+}}{\left(\partial X^{+}\right)^{5}}\right)(\mathbf{z}')\\
&\quad+\frac{\theta-\theta'}{\mathbf{z}-\mathbf{z}'}\left(-\frac{\partial^{2}DX^{+}}{2\left(\partial X^{+}\right)^{3}}+\frac{3\partial^{2}X^{+}\partial DX^{+}}{2\left(\partial X^{+}\right)^{4}}-\frac{\partial^{3}X^{+}DX^{+}}{2\left(\partial X^{+}\right)^{4}}\right.\\
&\qquad\left.+\frac{\left(\partial^{2}X^{+}\right)^{2}DX^{+}}{\left(\partial X^{+}\right)^{5}}-\frac{\partial^{2}DX^{+}\partial DX^{+}DX^{+}}{\left(\partial X^{+}\right)^{5}}\right)(\mathbf{z}')\Bigg]
\end{aligned}
$$

$X^{\pm}$ CFT ▸ BACK

From these, it is possible to prove

$$T(\mathbf{z})\,T(\mathbf{z}') \sim \frac{12-d}{4\left(\mathbf{z}-\mathbf{z}'\right)^3} + \frac{\theta-\theta'}{\left(\mathbf{z}-\mathbf{z}'\right)^2}\frac{3}{2}T(\mathbf{z}') + \frac{1}{\mathbf{z}-\mathbf{z}'}\frac{1}{2}DT(\mathbf{z}') + \frac{\theta-\theta'}{\mathbf{z}-\mathbf{z}'}\partial T(\mathbf{z}')$$

$T(\mathbf{z})$ satisfies the super Virasoro algebra with $\hat{c} = 12 - d$.

Bosonization BACK

$$
\begin{aligned}
\beta(z) &= e^{-\phi}\partial\xi(z) \\
\gamma(z) &= \eta e^{\phi}(z) \\
\delta(\beta) &= e^{\phi} \\
\delta(\gamma) &= e^{-\phi}
\end{aligned}
$$

Superghost correation function BACK

Verlinde-Verlinde

$$\left\langle \prod_i e^{\phi}(z_i) \prod_r e^{-\phi}(Z_r) \right\rangle$$

$$\propto \frac{1}{\vartheta[\alpha]\left(\sum z_i - \sum Z_r - 2\triangle\right)} \cdot \frac{\prod_{i,r} E(z_i, Z_r)}{\prod_{i>j} E(z_i, z_j) \prod_{r>s} E(Z_r, Z_s)} \cdot \frac{\prod_r \sigma(Z_r)^2}{\prod_i \sigma(z_i)^2}$$

$\vartheta[\alpha]\left(\sum z_i - \sum Z_r - 2\triangle\right)$: Riemann's theta function

$\sigma(z)$: a holomorphic $\frac{g}{2}$ form

Backup

Amplitudes from the first-quantized formalism

$$
\begin{aligned}
A &= \sum_{\text{worldsheet}} \int \frac{[dg_{mn} d\chi_a dX^\mu d\psi^\mu]}{\text{superrep.} \times \text{superWeyl}} e^{-I} V_1 \cdots V_N \\
&= \sum_{\text{worldsheet}} \int \prod_\alpha dm_\alpha \prod_\sigma d\eta_\sigma \, [dX^\mu d\psi^\mu dbdcd\beta d\gamma] \\
&\qquad \times e^{-I_{\text{g.f.}}} V_1 \cdots V_N \prod_\alpha B_\alpha \prod \delta(\beta_\sigma)
\end{aligned}
$$

- $\frac{\text{space of } g_{mn}, \chi_a}{\text{superrep.} \times \text{superWeyl}}$ =supermoduli space of superRiemann surface
- m_α,η_σ: coordinates of the supermoduli space GO
- $B_\alpha, \delta(\beta_\sigma)$: antighost insertions to soak up the zero modes

$$
\beta_\sigma = \int d^2 z \frac{\partial \chi_{\bar{z}}^{\theta \text{rep.}}}{\partial \eta_\sigma} \beta
$$

Picture changing operator

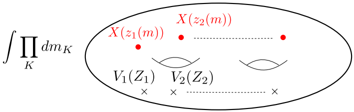
Verlinde-Verlinde

If one takes η_σ so that $\frac{\partial \chi_{\bar{z}}^{\theta \text{rep.}}}{\partial \eta_\sigma} = \delta^2 (z - z_\sigma)$ and integrating over η_σ we get

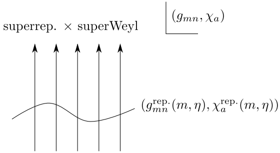
$$\begin{aligned} A &= \sum_{\text{worldsheet}} \int \prod_\alpha dm_\alpha \prod_\sigma d\eta_\sigma \left[dX^\mu d\psi^\mu dbdcd\beta d\gamma \right] \\ &\qquad \times e^{-I_{\text{g.f.}}} V_1 \cdots V_N \prod_\alpha B_\alpha \prod_\sigma \delta(\beta_\sigma) \\ &= \sum_{\text{worldsheet}} \int \prod_\alpha dm_\alpha \left[dX^\mu d\psi^\mu dbdcd\beta d\gamma \right] \\ &\qquad \times e^{-I} V_1 \cdots V_N \prod_\alpha B_\alpha \prod_\sigma X(z_\sigma) \end{aligned}$$

Picture changing operators BACK

- Taking $\frac{\partial \chi_{\bar{z}}^{\theta\text{rep.}}}{\partial \eta_\sigma} = \delta^2(z - z_\sigma)$ we get the amplitudes with picture changing operators inserted.



- We can freely take z_σ as long as $\frac{\partial \chi_{\bar{z}}^{\theta\text{rep.}}}{\partial \eta_\sigma}$ $(\sigma = 1, \cdots 2g - 2 + N)$ span the space transverse to the symmetry orbits. It is a "gauge choice".



First-quantized formalism BACK

Worldsheet action

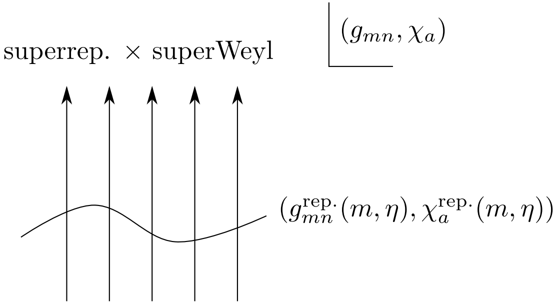
$$I = \frac{1}{8\pi}\int d^2\sigma\sqrt{g}\,[g^{mn}\partial_m X^\mu \partial_n X_\mu - i\psi^\mu\gamma^m\partial_m\psi_\mu - \psi^\mu\gamma^a\gamma^m\chi_a\partial_m X_\mu + \frac{1}{4}\left(\psi^\mu\gamma^a\gamma^b\chi_a\right)\chi_b\psi_\mu\Big]$$

- χ_a: gravitino field on the worlsheet
- superreparametrization invariance and super Weyl invariance

Amplitude

$$A = \sum_{\text{worldsheet}}\int \frac{[dg_{mn}d\chi_a dX^\mu d\psi^\mu]}{\text{superrep.}\times\text{superWeyl}} e^{-I} V_1\cdots V_N$$

First-quantized formalism



η_σ: odd moduli (Grassmann odd) BACK

Picture changing operator

A convenient choice is $\chi_{\bar{z}}^{(\sigma)\theta} = \delta^2(z - z_\sigma)$ and $\chi_{\bar{z}}^{\theta} = \sum_\sigma \eta_\sigma \delta^2(z - z_\sigma)$

$$\beta_\sigma = \int d^2 z \chi_{\bar{z}}^{(\sigma)\theta} \beta_{z\theta} = \beta(z_\sigma)$$

$$I_{\text{g.f.}} = \cdots + \int d^2 z \chi_{\bar{z}}^{\theta} G = I' + \sum_\sigma \eta_\sigma G(z_\sigma)$$

BACK

$$\int \prod_\alpha dm_\alpha \prod_\sigma d\eta_\sigma \left[dX^\mu d\psi^\mu dbdcd\beta d\gamma\right]$$
$$\times e^{-I_{\text{g.f.}}} V_1 \cdots V_N \prod_\alpha B_\alpha \prod_\sigma \delta(\beta_\sigma)$$

Picture changing operator

A convenient choice is $\chi_{\bar{z}}^{(\sigma)\theta} = \delta^2(z - z_\sigma)$ and $\chi_{\bar{z}}^{\theta} = \sum_\sigma \eta_\sigma \delta^2(z - z_\sigma)$

$$\begin{aligned} \beta_\sigma &= \int d^2 z \chi_{\bar{z}}^{(\sigma)\theta} \beta_{z\theta} = \beta(z_\sigma) \\ I_{\mathrm{g.f.}} &= \cdots + \int d^2 z \chi_{\bar{z}}^{\theta} G = I' + \sum_\sigma \eta_\sigma G(z_\sigma) \end{aligned}$$

BACK

$$\int \prod_\alpha dm_\alpha \prod_\sigma d\eta_\sigma \left[dX^\mu d\psi^\mu db dc d\beta d\gamma\right] \times e^{-I_{\mathrm{g.f.}}} V_1 \cdots V_N \prod_\alpha B_\alpha \prod_\sigma \delta(\beta_\sigma)$$

$$\propto \int \prod_\alpha dm_\alpha \left[dX^\mu d\psi^\mu db dc d\beta d\gamma\right] \times e^{-I'} V_1 \cdots V_N \prod_\alpha B_\alpha \prod_\sigma \left(\delta(\beta) G + \cdots\right)(z_\sigma)$$

Picture changing operator

A convenient choice is $\chi_{\bar{z}}^{(\sigma)\theta} = \delta^2(z - z_\sigma)$ and $\chi_{\bar{z}}^{\theta} = \sum_\sigma \eta_\sigma \delta^2(z - z_\sigma)$

$$\beta_\sigma = \int d^2 z \chi_{\bar{z}}^{(\sigma)\theta} \beta_{z\theta} = \beta(z_\sigma)$$

$$I_{\text{g.f.}} = \cdots + \int d^2 z \chi_{\bar{z}}^{\theta} G = I' + \sum_\sigma \eta_\sigma G(z_\sigma)$$

BACK

$$\int \prod_\alpha dm_\alpha \prod_\sigma d\eta_\sigma \left[dX^\mu d\psi^\mu dbdcd\beta d\gamma\right] \times e^{-I_{\text{g.f.}}} V_1 \cdots V_N \prod_\alpha B_\alpha \prod_\sigma \delta(\beta_\sigma)$$

$$\propto \int \prod_\alpha dm_\alpha \left[dX^\mu d\psi^\mu dbdcd\beta d\gamma\right] \times e^{-I'} V_1 \cdots V_N \prod_\alpha B_\alpha \prod_\sigma X(z_\sigma)$$