Mirzakhani-McShane identity and String field theory

Nobuyuki Ishibashi (University of Tsukuba)
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Superstring theory

- Superstring theory has been around for 40-50 years.
- In any theory of physics, there exists an equation (or action) from which everything can be deduced in principle.

Schrödinger equation $$i\hbar \frac{\partial}{\partial t}\psi = H\psi$$

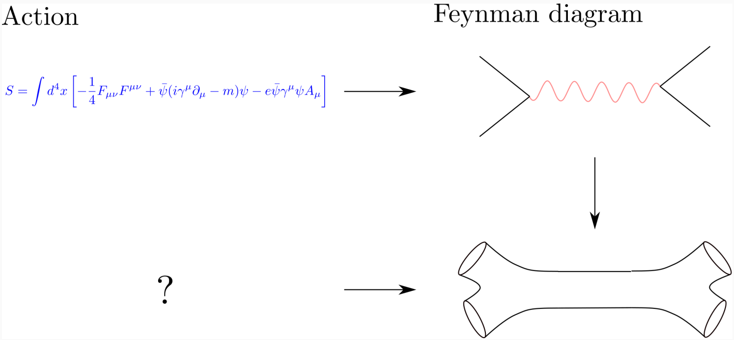
Einstein equation $$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}$$

$$\uparrow$$

$$S = \frac{c^3}{16\pi G} \int d^4x \sqrt{-g} R$$

- What is "the equation" (or action) for superstring theory?
- Unfortunately, a clear answer to this question is not known yet.

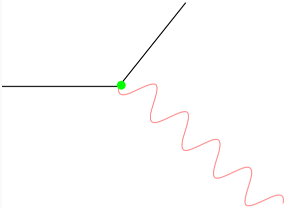
What is the equation or action?



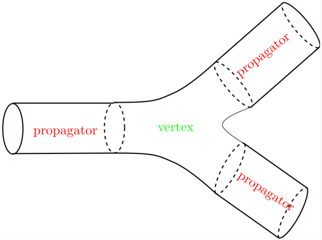
- An honest approach to this question is given by string field theory (SFT).

Strategy

$$S = \int d^4x \left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - e\bar{\psi}\gamma^\mu\psi A_\mu\right]$$

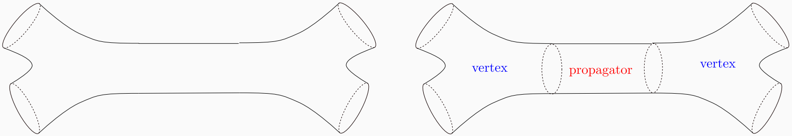


$$S = \Phi K \Phi + \Phi^3$$



- Once we are able to identify the propagators and vertices in the Feynman diagrams of string theory, we can construct the action.

SFT



- The amplitudes in string theory are expressed by Feynman diagrams = worldsheets~Riemann surfaces
- In an SFT, the worldsheets appear by combining propagators and vertices.
- In order to construct an SFT, we should define a rule to decompose all the worldsheets into propagators and vertices systematically.
    - In general, we need infinitely many vertices to do so.

$$S = \Phi K \Phi + \Phi^3 + \Phi^4 + \cdots + \hbar \Phi + \cdots$$

Status of SFT

- Bosonic strings
  - There exist SFT's with actions as simple as

$$S = \Phi K \Phi + \Phi^3$$

    - Light-cone gauge SFT(Kaku-Kikkawa), ($\alpha = p^+$) HIKKO (Hata-Itoh-Kugo-Kunitomo-Ogawa), covariantized light-cone
    - Witten's SFT
- Superstrings
  - If one tries to formulate SFT for superstrings generalizing the theories above, one runs into the "spurious singularity" problem.
  - Sen constructed an action avoiding this problem with the form

$$S = \Phi K \Phi + \Phi^3 + \Phi^4 + \cdots + \hbar \Phi + \cdots$$

    The terms in the action are not known in closed forms in general.

- Most of the string theorists believe that superstring theory can be described by some gauge theories or matrix models, assuming AdS/CFT correspondence or other dualities.

This talk

- It may be helpful to find out yet another rule to decompose Riemann surfaces such that the SFT becomes simple.

$$S = \Phi K \Phi + \Phi^3$$

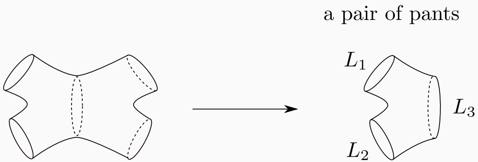
  - Having such a theory would be useful if one tries to prove AdS/CFT correspondence.

- In this talk, we would like to construct an SFT for bosonic strings based on the so-called pants decomposition of hyperbolic surfaces.
  - It is known that there exists a problem in constructing such a theory.
  - We overcome the problem using the Mirzakhani-McShane identity. PTEP 023B05(2023)

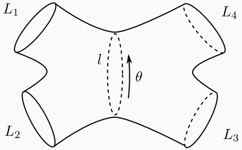
Plan of the talk

1. Pants decomposition
2. Mirzakhani recursion
3. A recursion relation for the off-shell amplitudes of closed bosonic strings
4. The Fokker-Planck formalism
5. BRST invariant formulation
6. Conclusions

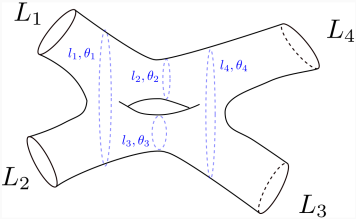
1. Pants decomposition



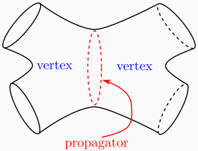
- A Riemann surface with $2g-2+n>0$ admits a unique hyperbolic metric such that the boundaries are geodesics.
- It can be decomposed into pairs of pants whose boundaries are geodesics.
  - The shape of a pair of pants is uniquely fixed by the lengths of the boundaries.
  - The shape of the surface can be described by l, θ



Pants decomposition



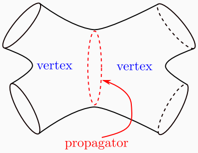
- In general, the moduli space of Riemann surfaces (~space of the shape of the surfaces) can be parametrized by the lengths and twist angles in a pants decomposition.



- This fact implies that we may be able to construct an SFT with

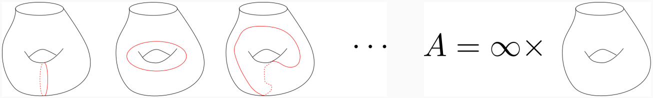
$$S = \Phi K \Phi + \Phi^3$$

An SFT based on the pants decomposition?



$$S = \Phi K \Phi + \Phi^3$$

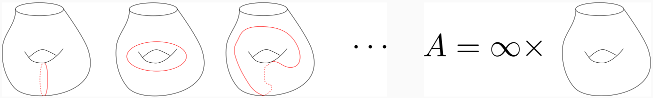
- This action does not work. (D'Hoker-Gross)
  - One-loop one point amplitudes diverge because the pants decomposition is not unique.



$$A = \int dl d\theta \langle BV \rangle = \infty$$

  - Most of the amplitudes diverge in the same way.

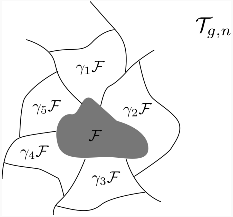
$$A = \int \prod_{s=1}^{3g-3+n} (dl_s d\theta_s) \langle BV_1 \cdots V_n \rangle = \int_{\mathcal{T}_{g,n}} \langle BV_1 \cdots V_n \rangle = \infty$$



- These different pants decompositions are transformed to each other by diffeomorphisms not isotopic to identity.
- The group of such diffeomorphisms is called the mapping class group.
  - The amplitudes are invariant under the action of the group (modular invariance).

$$A = \int_{\mathcal{T}_{g,n}} \langle B V_1 \cdots V_n \rangle = \infty \times \int_{\mathcal{F}} \langle B V_1 \cdots V_n \rangle$$

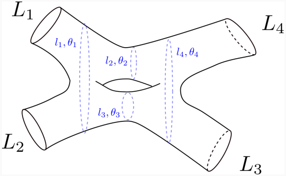
$\mathcal{F} = \mathcal{T}_{g,n}/\mathrm{Mod}_{g,n}$: fundamental domain



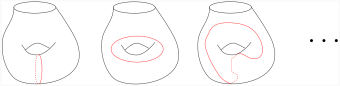
2. Mirzakhani recursion

- The volume of the moduli space of Riemann surfaces with genus g and n boundaries whose lengths are $L_1, \cdots, L_n$ is given by

$$V_{g,n}(L_1, \cdots, L_n) = \int \prod_{s=1}^{3g-3+n} \left[\frac{l_s dl_s d\theta_s}{2\pi} \right]$$

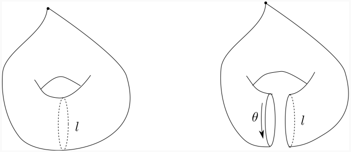


- Integrating over $0 < l_s < \infty$, the integral diverges.



$$\int_0^\infty \int_0^{2\pi} \frac{l dl d\theta}{2\pi} = \infty \times$$

- We should integrate over the fundamental domain $\mathcal{F}$, which is very complicated in general.



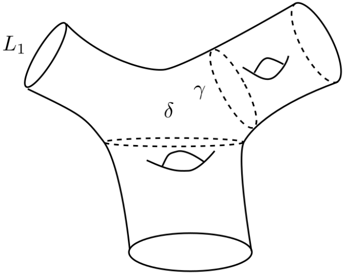
- McShane identity (1998): for $f(l) = \frac{2}{1+e^l}$
$$1 = \sum_{\gamma \in \mathbf{Mod}_{1,1}} f(\gamma \cdot l)$$
- $V_{1,1}$ can be calculated multiplying this by $\int_{\mathcal{F}} \frac{l dl d\theta}{2\pi}$ (Mirzakhani)

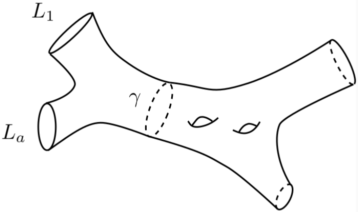
$$\begin{aligned} V_{1,1}(0) &= \int_{\mathcal{F}} \frac{l dl d\theta}{2\pi} = \int_{\mathcal{F}} \sum_{\gamma} f(\gamma \cdot l) \frac{l dl d\theta}{2\pi} \\ &= \int_{\mathcal{F}} \sum_{\gamma} f(\gamma \cdot l) \frac{\gamma \cdot l d(\gamma \cdot l) d(\gamma \cdot \theta)}{2\pi} = \sum_{\gamma} \int_{\gamma \mathcal{F}} f(l) \frac{l dl d\theta}{2\pi} \\ &= \int \frac{dl d\theta l}{2\pi} \frac{2}{1+e^l} = \frac{\pi^2}{6} \end{aligned}$$

Mirzakhani-McShane identity

- Mirzakhani obtained identities for general g, n with $2g-2+n>0$.

$$L_1 = \sum_{\{\gamma,\delta\}\in\mathcal{C}_1} D_{L_1 l_\gamma l_\delta} + \sum_{a=2}^{n} \sum_{\gamma\in\mathcal{C}_a} \left(T_{L_1 L_a l_\gamma} + D_{L_1 L_a l_\gamma}\right)$$





$$D_{LL'L''} = 2\left(\log(e^{\frac{L}{2}} + e^{\frac{L'+L''}{2}}) - \log(e^{-\frac{L}{2}} + e^{\frac{L'+L''}{2}})\right)$$

$$T_{LL'L''} = \log \frac{\cosh\frac{L''}{2} + \cosh\frac{L+L'}{2}}{\cosh\frac{L''}{2} + \cosh\frac{L-L'}{2}}$$

Mirzakhani recursion relation

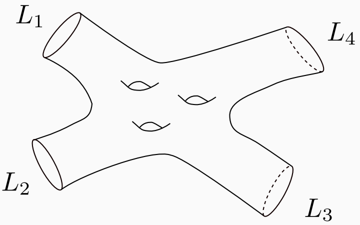
- Multiplying the generalized McShane identity by $\int_{\mathcal{F}} \prod_s \left[\frac{l_s dl_s d\theta_s}{2\pi} \right]$, we get

$$\begin{aligned} LV_{g,n+1}(L,\mathbf{L}) &= \frac{1}{2}\int_0^\infty dL'L' \int_0^\infty dL''L'' D_{LL'L''} V_{g-1,n+2}(L',L'',\mathbf{L}) \\ &+\frac{1}{2}\int_0^\infty dL'L' \int_0^\infty dL''L'' D_{LL'L''} \sum_{\text{stable}} V_{g_1,n_1}(L',\mathbf{L}_1) V_{g_2,n_2}(L'',\mathbf{L}_2) \\ &+\sum_{a=1}^{n} \int_0^\infty dL'L' \left(T_{L_1 L_a L'} + D_{L_1 L_a L'}\right) V_{g,n}(L,\mathbf{L}\backslash L_a) \end{aligned}$$

- $V_{g,n+1}(l,\mathbf{L})$ can be expressed by the volumes for simpler surfaces.
- One can calculate $V_{g,n}(L_1,\cdots,L_n)$ by solving this equation.

3. A recursion relation for the off-shell amplitudes of closed bosonic strings

2. A recursion relation for the off-shell amplitudes of closed bosonic strings



- In string theory, the amplitudes are given by integrals over the moduli space of Riemann surfaces

$$A_{g,n}^{i_1\cdots i_n} = \int_{\mathcal{F}} \prod_s [dl_s d\theta_s] \langle \prod_s [b(\partial_{l_s}) b(\partial_{\theta_s})] V_{i_1} \cdots V_{i_n} \rangle_{\Sigma_{g,n,L_1,\cdots,L_n}}$$

- It is conceivable that we can derive a recursion relation for these amplitudes in the same way as we did for the recursion relation for

$$V_{g,n}(L_1, \cdots, L_n) = \int_{\mathcal{F}} \prod_s \left[\frac{l_s dl_s d\theta_s}{2\pi} \right]$$

The recursion relation

generalized McShane identity

$$L_1 = \sum_{\{\gamma,\delta\}\in\mathcal{C}_1} D_{L_1 l_\gamma l_\delta} + \sum_{a=2}^{n}\sum_{\gamma\in\mathcal{C}_a}\left(T_{L_1 L_a l_\gamma} + D_{L_1 L_a l_\gamma}\right)$$

$$\xrightarrow{\int_{\mathcal{F}}\prod_s\left[\frac{l_s dl_s d\theta_s}{2\pi}\right]\times}$$

recursion relation for

$$V_{g,n}(L_1,\cdots,L_n) = \int_{\mathcal{F}}\prod_s\left[\frac{l_s dl_s d\theta_s}{2\pi}\right]$$

$$\Big\downarrow\ \int_{\mathcal{F}}\prod_s dl_s d\theta_s\left\langle\prod_s\left[b(\partial_{l_s})b(\partial_{\theta_s})\right]V_{i_1}\cdots V_{i_n}\right\rangle\times$$

recursion relation for

$$A^{i_1\cdots i_n}_{g,n} = \int_{\mathcal{F}}\prod_s dl_s d\theta_s\left\langle\prod_s\left[b(\partial_{l_s})b(\partial_{\theta_s})\right]V_{i_1}\cdots V_{i_n}\right\rangle$$

$$\begin{aligned}
L_1 A^{I_1\cdots I_n}_{g,n} &= L_1 G^{I_1 I_2}\delta_{g,0}\delta_{n,2} \\
&+\frac{1}{2}D^{I_1 J' J}G_{JI}G_{J'I'}\left[A^{II' I_2\cdots I_n}_{g-1,n+1} + {\sum}'\frac{\varepsilon_{\mathcal{I}_1\mathcal{I}_2}}{(n_1-1)!(n_2-1)!}A^{I\mathcal{I}_1}_{g_1,n_1}A^{I'\mathcal{I}_2}_{g_2,n_2}\right] \\
&+\sum_{a=2}^{n}\varepsilon_a T^{I_1 I_a J}G_{JI}A^{II_2\cdots\hat{I}_a\cdots I_n}_{g,n-1}
\end{aligned}$$

$$A_{g,n}^{I_1\cdots I_n} = \int_{\mathcal{F}} \prod_s [dl_s d\theta_s] \langle \prod_s [b(\partial_{l_s}) b(\partial_{\theta_s})] B_{\alpha_1}\cdots B_{\alpha_n} V_{i_1}\cdots V_{i_n}\rangle$$

$$B_{\alpha_a} \equiv \begin{cases} 1 & \alpha_a = + \\ (b_0^{(a)} - \bar{b}_0^{(a)}) b_{S_a}(\partial_{L_a}) \int_0^{2\pi} \frac{d\theta_a}{2\pi} e^{i\theta_a (L_0^{(a)} - \bar{L}_0^{(a)})} & \alpha_a = - \end{cases}$$

$$\begin{aligned} L_1 A_{g,n}^{I_1\cdots I_n} \quad = \quad & L_1 G^{I_1 I_2}\delta_{g,0}\delta_{n,2} \\ & + \frac{1}{2} D^{I_1 J' J} G_{JI} G_{J'I'} \left[A_{g-1,n+1}^{II' I_2\cdots I_n} + \sum{}' \frac{\varepsilon_{\mathcal{I}_1\mathcal{I}_2}}{(n_1-1)!(n_2-1)!} A_{g_1,n_1}^{I\mathcal{I}_1} A_{g_2,n_2}^{I'\mathcal{I}_2} \right] \\ & + \sum_{a=2}^{n} \varepsilon_a T^{I_1 I_a J} G_{JI} A_{g,n-1}^{II_2\cdots \hat{I}_a \cdots I_n} \end{aligned}$$

$$\begin{aligned} T^{I_1 I_2 I_3} \quad & \equiv \quad T_{L_1 L_2 L_3} \langle B_{\alpha_1} B_{\alpha_2} B_{\alpha_3} V^{i_1} V^{i_2} V^{i_3} \rangle \\ D^{I_1 I_2 I_3} \quad & \equiv \quad D_{L_1 L_2 L_3} \langle B_{\alpha_1} B_{\alpha_2} B_{\alpha_3} V^{i_1} V^{i_2} V^{i_3} \rangle \\ G_{I_1 I_2} \quad & \equiv \quad \langle \varphi_{i_1}^c | \varphi_{i_2}^c \rangle (-1)^{n_{\varphi_{i_2}}} \delta(L_1 - L_2) \delta_{\alpha_1, -\alpha_2} \, , \end{aligned}$$

4. The Fokker-Planck formalism

$$\begin{aligned} L_1 A_{g,n}^{I_1\cdots I_n} &= L_1 G^{I_1 I_2}\delta_{g,0}\delta_{n,2} \\ &+\frac{1}{2}D^{I_1 J' J}G_{JI}G_{J'I'}\left[A_{g-1,n+1}^{II' I_2\cdots I_n}+\sum{}' \frac{\varepsilon_{\mathcal{I}_1\mathcal{I}_2}}{(n_1-1)!(n_2-1)!}A_{g_1,n_1}^{I\mathcal{I}_1}A_{g_2,n_2}^{I'\mathcal{I}_2}\right] \\ &+\sum_{a=2}^{n}\varepsilon_a T^{I_1 I_a J}G_{JI}A_{g,n-1}^{II_2\cdots\hat{I}_a\cdots I_n} \end{aligned}$$

- One can derive the amplitudes $A_{g,n}^{I_1\cdots I_n}$ perturbatively solving this equation.
  - This equation can be regarded as the Schwinger-Dyson equation of the string theory.
  - We may be able to construct an SFT from this equation.

- This equation can be turned into an SFT via the method developed by Kawai-NI, Jevicki-Rodrigues, Ikehara-Kawai-Mogami-Nakayama-Sasakura-NI , Ikehara, ...

The Fokker-Planck formalism

- Euclidean field theory

$$\langle \phi(x_1)\cdots\phi(x_n)\rangle = \frac{\int [d\phi] e^{-S[\phi]} \phi(x_1)\cdots\phi(x_n)}{\int [d\phi] e^{-S[\phi]}}$$

- Fokker-Planck formalism

$$\langle \phi(x_1)\cdots\phi(x_n)\rangle = \lim_{\tau\to\infty} \langle 0| e^{-\tau \hat{H}_{\mathrm{FP}}} \hat{\phi}(x_1)\cdots\hat{\phi}(x_n)|0\rangle$$

$$[\hat{\pi}(x), \hat{\phi}(y)] = \delta(x-y)\,, [\hat{\pi},\hat{\pi}] = [\hat{\phi},\hat{\phi}] = 0$$

$$\langle 0|\hat{\phi}(x) = \hat{\pi}(x)|0\rangle = 0$$

$$\hat{H}_{\mathrm{FP}} = -\int dx \left(\hat{\pi}(x) + \frac{\delta S}{\delta \phi(x)}[\hat{\phi}] \right) \hat{\pi}(x)$$

- path integral

$$\langle \phi(x_1)\cdots\phi(x_n)\rangle = \frac{\int [d\pi d\phi] e^{-I_{\mathrm{FP}}} \phi(0,x_1)\cdots\phi(0,x_n)}{\int [d\pi d\phi] e^{-I_{\mathrm{FP}}}}$$

$$I_{\mathrm{FP}} = \int_0^\infty d\tau \left[-\int dx \pi \partial_\tau \phi + H_{\mathrm{FP}} \right]$$

The Fokker-Planck formalism for closed bosonic strings

- Following the procedure, we obtain the FP formalism
  - Correlation functions

$$\langle\!\langle \phi^{I_1} \cdots \phi^{I_n} \rangle\!\rangle^{\mathrm{c}} = \sum_{g=0}^{\infty} g_{\mathrm{s}}^{2g-2+n} A_{g,n}^{I_1 \cdots I_n}$$

  - The FP formalism

$$\langle\!\langle \phi^{I_1} \cdots \phi^{I_n} \rangle\!\rangle = \lim_{\tau \to \infty} \langle\!\langle 0 | e^{-\tau \hat{H}} \hat{\phi}^{I_1} \cdots \hat{\phi}^{I_n} | 0 \rangle\!\rangle$$

$$[\hat{\pi}_I, \hat{\phi}^K] = \delta_I{}^K$$
$$[\hat{\pi}_I, \hat{\pi}_K] = [\hat{\phi}^I, \hat{\phi}^K] = 0$$
$$\langle\!\langle 0 | \hat{\phi}^I = \hat{\pi}_I | 0 \rangle\!\rangle = 0$$

$$\begin{aligned} \hat{H} \quad = \quad & -L \hat{\pi}_I \hat{\pi}_{I'} G^{I'I} + L \hat{\phi}^I \hat{\pi}_I \\ & -\frac{1}{2} g_{\mathrm{s}} D^{II'I''} G_{I''K''} G_{I'K'} \hat{\phi}^{K''} \hat{\phi}^{K'} \hat{\pi}_I \\ & -g_{\mathrm{s}} T^{II'I''} G_{I''K''} \hat{\phi}^{K''} \hat{\pi}_{I'} \hat{\pi}_I \end{aligned}$$

  - The Hamiltonian consists of kinetic terms and three string interaction terms.

The action

- It is possible to (formally) define the action $S[\phi]$.

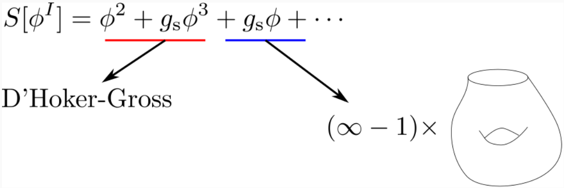
$$\frac{e^{-S[\phi]}}{\int [d\phi] e^{-S[\phi]}} = \lim_{\tau\to\infty} \langle\!\langle 0 | e^{-\tau \hat{H}} \delta(\phi - \hat{\phi}) | 0 \rangle\!\rangle$$

$$\begin{aligned}
&\frac{\int [d\phi] e^{-S[\phi]} \phi^{I_1} \dots \phi^{I_n}}{\int [d\phi] e^{-S[\phi]}} \\
&= \lim_{\tau\to\infty} \langle\!\langle 0 | e^{-\tau \hat{H}} \int [d\phi] \delta(\phi - \hat{\phi}) \phi^{I_1} \dots \phi^{I_n} | 0 \rangle\!\rangle \\
&= \lim_{\tau\to\infty} \langle\!\langle 0 | e^{-\tau \hat{H}} \hat{\phi}^{I_1} \dots \hat{\phi}^{I_n} | 0 \rangle\!\rangle
\end{aligned}$$

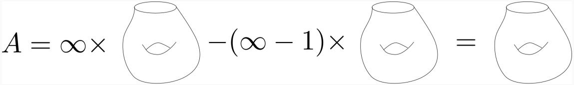
- One can calculate $S[\phi^I]$ perturbatively.

$$\begin{aligned}
S[\phi^I] \quad = \quad & \frac{1}{2} G_{IJ} \phi^I \phi^J - \frac{g_s}{6} A_{0,3}^{II'I''} G_{IJ} G_{I'J'} G_{I''J''} \phi^{J''} \phi^{J'} \phi^J \\
& + \frac{g_s}{L} T^{II'I''} G_{I'I''} G_{IJ} \phi^J + \mathcal{O}(g_s^2)
\end{aligned}$$

The action



- $S[\phi^I]$ is divergent and ill defined.
    - The 1 loop 1 point amplitude



    - $S[\phi^I]$ includes infinitely many divergent counterterms.

5. BRST invariant formulation

BRST symmetry on the worldsheet

- We need the worldsheet BRST symmetry to define the physical states with positive norm.

$$\begin{aligned} Q|\mathrm{phys.}\rangle &= 0 \\ |\ \rangle &\sim |\ \rangle + Q|\ \rangle' \end{aligned}$$

- In order to discuss this symmetry, we change the notation

$$\begin{aligned} |\phi^\alpha(L)\rangle &\equiv \sum_i \hat{\phi}^I |\varphi_i^c\rangle \\ |\pi_\alpha(L)\rangle &\equiv \sum_i |\varphi_i\rangle \hat{\pi}_I \end{aligned}$$

$$\begin{aligned} \hat{H} = & \int_0^\infty dL L \left[\langle R|\phi^\alpha(L)\rangle|\pi_\alpha(L)\rangle - \langle R|\pi_\alpha(L)\rangle|\pi_{-\alpha}(L)\rangle\right] \\ & - g_s \int dL_1 dL_2 dL_3 \langle T_{L_2 L_3 L_1}|B^1_{-\alpha_1} B^2_{\alpha_2} B^3_{\alpha_3}|\phi^{\alpha_1}(L_1)\rangle_1|\pi_{\alpha_2}(L_2)\rangle_2|\pi_{\alpha_3}(L_3)\rangle_3 \\ & - \frac{1}{2} g_s \int dL_1 dL_2 dL_3 \langle D_{L_3 L_1 L_2}|B^1_{-\alpha_1} B^2_{-\alpha_2} B^3_{\alpha_3}|\phi^{\alpha_1}(L_1)\rangle_1|\phi^{\alpha_2}(L_2)\rangle_2|\pi_{\alpha_3}(L_3)\rangle_3 \end{aligned}$$

  - The BRST transformation

$$\begin{aligned} &\delta_\epsilon|\phi^+(L)\rangle = \epsilon P_- Q|\phi^+(L)\rangle & &\delta_\epsilon|\pi_+(L)\rangle = \epsilon Q|\pi_+(L)\rangle - \epsilon b_0^- P\partial_L|\pi_-(L)\rangle \\ &\delta_\epsilon|\phi^-(L)\rangle = \epsilon Q|\phi^-(L)\rangle - \epsilon b_0^- P\partial_L|\phi^+(L)\rangle & &\delta_\epsilon|\pi_-(L)\rangle = \epsilon P_- Q|\pi_-(L)\rangle \end{aligned}$$

$\hat{H}$ is not BRST invariant

- $\hat{H}$ is not BRST invariant.
  - If it were, FP formalism would be modular invariant
  - Let $\hat{Q}$ be the generator of the BRST transformation

$$\delta\hat{H} = [\hat{Q}, \hat{H}] = \int_0^\infty dL \left(\langle R|\mathcal{Q}^\alpha(L)\rangle|\pi_\alpha(L)\rangle + \langle R|\mathcal{T}^\alpha(L)\rangle[\hat{Q}, |\pi_\alpha(L)\rangle] \right)$$

$$\hat{H} = \int_0^\infty dL \langle R|\mathcal{T}^\alpha(L)\rangle|\pi_\alpha(L)\rangle$$

$$|\mathcal{Q}^\alpha(L)\rangle \equiv [\hat{Q}, |\mathcal{T}^\alpha(L)\rangle]$$

- The amplitudes are invariant, because $|\mathcal{Q}^\alpha(L)\rangle, |\mathcal{T}^\alpha(L)\rangle$ are "null quantities" satisfying

$$\left[\lim_{\tau\to\infty} \langle\langle 0|e^{-\tau\hat{H}}\right] |\mathcal{T}^\alpha(L)\rangle = 0$$

$$\left[\lim_{\tau\to\infty} \langle\langle 0|e^{-\tau\hat{H}}\right] |\mathcal{Q}^\alpha(L)\rangle = 0$$

BRST invariant formulation

- We can modify the Hamiltonian by introducing the auxiliary fields $|\lambda^{\mathcal{Q}}_\alpha(L)\rangle, |\lambda^{\mathcal{T}}_\alpha(L)\rangle$ so that it becomes BRST invariant and still yields the correct amplitudes.

$$\hat{H} \to \hat{H} + \int_0^\infty dL \left(\langle R|\mathcal{Q}^\alpha(L)\rangle|\lambda^{\mathcal{Q}}_\alpha(L)\rangle + \langle R|\mathcal{T}^\alpha(L)\rangle|\lambda^{\mathcal{T}}_\alpha(L)\rangle \right)$$

$$\delta\hat{H} = \int_0^\infty dL \left(\langle R|\mathcal{Q}^\alpha(L)\rangle|\pi_\alpha(L)\rangle + \langle R|\mathcal{T}^\alpha(L)\rangle[\hat{Q}, |\pi_\alpha(L)\rangle] \right)$$

- The action

$$I_{\mathrm{FP}} = \int_0^\infty d\tau \Big[-\int_0^\infty dL \langle R|\pi_\alpha(\tau,L)\rangle \frac{\partial}{\partial\tau}|\phi^\alpha(\tau,L)\rangle + H(\tau) + \int_0^\infty dL \left(\langle R|\mathcal{Q}^\alpha(\tau,L)\rangle|\lambda^{\mathcal{Q}}_\alpha(\tau,L)\rangle + \langle R|\mathcal{T}^\alpha(\tau,L)\rangle|\lambda^{\mathcal{T}}_\alpha(\tau,L)\rangle \right) \Big]$$

  - This action is invariant under the BRST transformation.
  - It consists of kinetic terms and three string interaction terms.

6. Conclusions

5. Conclusions

$$I_{\mathrm{FP}} = \int_0^\infty d\tau \Big[-\int_0^\infty dL \langle R|\pi_\alpha(\tau,L)\rangle \frac{\partial}{\partial\tau}|\phi^\alpha(\tau,L)\rangle + H(\tau) + \int_0^\infty dL \Big(\langle R|\mathcal{Q}^\alpha(\tau,L)\rangle|\lambda_\alpha^{\mathcal{Q}}(\tau,L)\rangle + \langle R|\mathcal{T}^\alpha(\tau,L)\rangle|\lambda_\alpha^{\mathcal{T}}(\tau,L)\rangle \Big) \Big]$$

- We have constructed an SFT for closed bosonic strings based on the pants decomposition via the Fokker-Planck formalism.
  - The action consists of kinetic terms and three string interaction terms.
  - It is manifestly invariant under a nilpotent BRST transformation and we can define the physical states using it.

- It is possible to construct a similar SFT using Strebel differentials and combinatorial moduli space. (N.I. PTEP 2024 (2024) 7, 073B02)
- The technique here can be used to construct classical solutions of SFT . Firat-Valdes-Meller
- SFT for superstrings?