Strebel differentials and string field theory

JPS meeting at Hokkaido University

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Sep. 17, 2024

Quadratic differentials

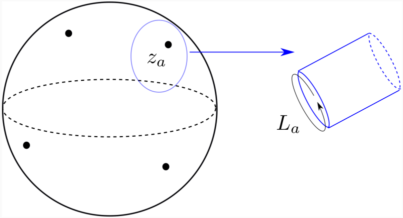
- On a punctured Riemann surface (~Feynman diagram of string theory), let us consider a quadratic differential $\phi(z)dz^2$ such that
  - near punctures ($z \sim z_a$ $(a = 1, \cdots n)$)

$$\phi(z)dz^2 \sim -\left(\frac{L_a}{2\pi}\right)^2 \frac{dz^2}{(z - z_a)^2}$$

  with $L_a > 0$ and holomorphic for $z \neq z_a$

- One can define a locally flat metric

$$ds^2 = |\phi(z)|\, dz d\bar{z} = dw d\bar{w}$$
$$w = \int^z dz' \sqrt{\phi(z')}$$

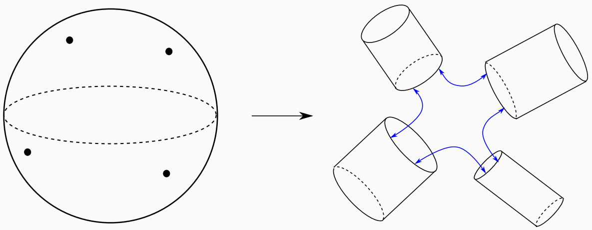


Strebel's theorem

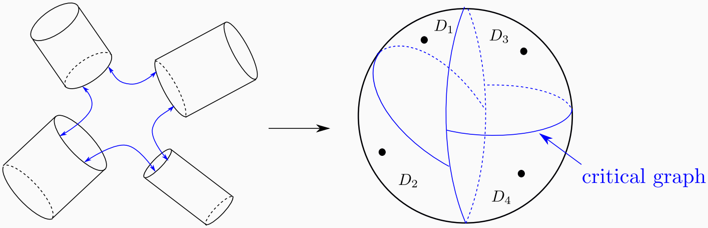
- Given a punctured Riemann surface and positive numbers $L_1, \cdots, L_n$, there exists the unique quadratic differential $\phi(z)dz^2$ (Strebel differential) such that
  - for $z \sim z_a$, $\phi(z)dz^2 \sim -\left(\frac{L_a}{2\pi}\right)^2 \frac{dz^2}{(z-z_a)^2}$
  - holomorphic for $z \neq z_a$
  - with the metric

$$
\begin{aligned}
ds^2 &= |\phi(z)|\, dz d\bar{z} = dw d\bar{w} \\
& w = \int^z dz' \sqrt{\phi(z')}
\end{aligned}
$$

the surface looks like



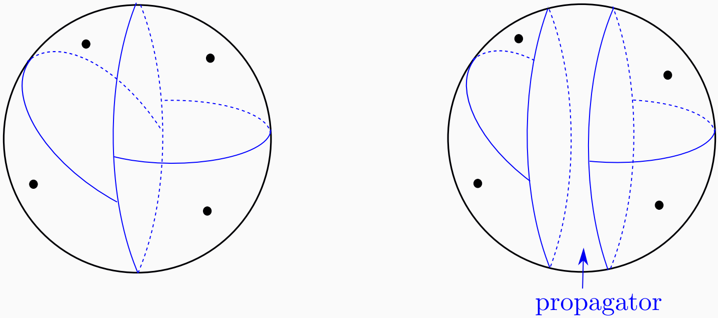
Strebel differentials

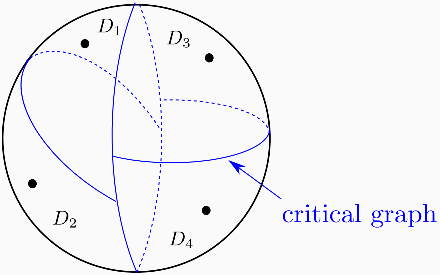


- To any punctured Riemann surface, there corresponds a graph called critical graph.
- Moduli spaces of punctured Riemann surfaces can be perametrized by the lengths of the edges of the critical graphs (combinatorial moduli space).
- Strebel differentials play important roles in
    - Kontsevich's proof of Witten conjecture
    - studying the free field limit of AdS/CFT (Gopakumar, ...)

Strebel differentials and string field theory

- Strebel differentials may be useful in constructing a string field theory.
- They were used to construct the interaction vertices of a closed bosonic string field theory in Saadi-Zwiebach, Kugo-Kunitomo-Suehiro, Kugo-Suehiro.
  - The nonpolynomial SFT reproduces the tree amplitudes.
  - Strebel's theorem and combinatorial moduli space are not relevant.





- If Strebel differentials are really important in describing the free field limit in AdS/CFT, it is worthwhile to construct an SFT using the combinatorial moduli space as basic tool.
    - We need to give up the conventional form of propagators.
- It is possible to construct such an SFT. (PTEP 2024 (2024) 7, 073B02)

1. Combinatorial pants decomposition

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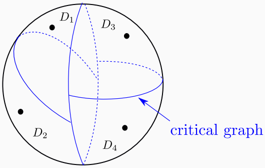
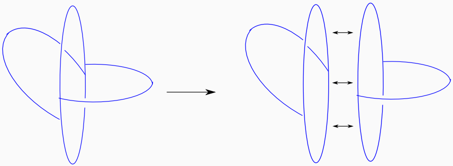
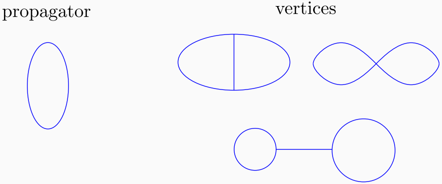
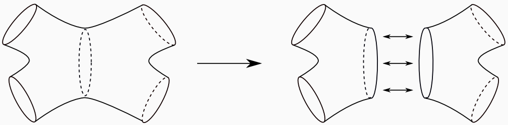




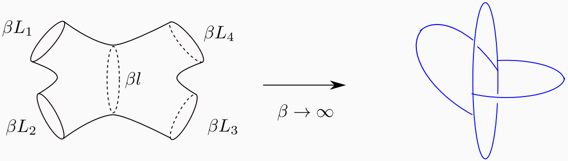
- The critical graphs can be decomposed into three string vertices. (combinatorial pants decomposition)
- We may be able to construct a theory with



Pants decomposition

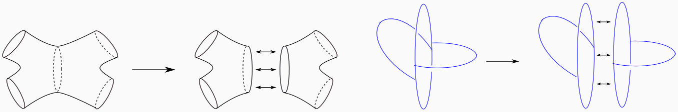


- A Riemann surface with boundaries ($2g - 2 + n > 0$) admits a hyperbolic metric ($R = -2$) such that the boundaries are geodesics.
  - It can be decomposed into pairs of pants whose boundaries are geodesics.
- The critical graphs in the combinatorial moduli space can be regarded as the long string limit of the hyperbolic surfaces. (Mondello, Do)



SFT

- The combinatorial pants decomposition can be considered as the long string limit of the hyperbolic pants decomposition.



- An SFT based on the hyperbolic pants decomposition was constructed (N.I.).
  - The construction is based on the Fenchel-Nielsen coordinates, Mirzkhani's recursion relation ...
- The combinatorial version of all these were defined in Andersen et al..
- It is possible to construct an SFT using these results.

2. The Fokker-Planck formalism

2. The Fokker-Planck formalism

In order to construct the theory, we need to employ the Fokker-Planck formalism

- Euclidean field theory

$$\langle \phi(x_1)\cdots\phi(x_n)\rangle = \frac{\int [d\phi] e^{-S[\phi]}\phi(x_1)\cdots\phi(x_n)}{\int [d\phi] e^{-S[\phi]}}$$

- Fokker-Planck formalism

$$\langle \phi(x_1)\cdots\phi(x_n)\rangle = \lim_{\tau\to\infty}\langle 0|e^{-\tau \hat{H}_{\mathrm{FP}}}\hat{\phi}(x_1)\cdots\hat{\phi}(x_n)|0\rangle$$

$$[\hat{\pi}(x), \hat{\phi}(y)] = \delta(x-y)\,, [\hat{\pi}, \hat{\pi}] = [\hat{\phi}, \hat{\phi}] = 0$$

$$\langle 0|\hat{\phi}(x) = \hat{\pi}(x)|0\rangle = 0$$

$$\hat{H}_{\mathrm{FP}} = -\int dx \left(\hat{\pi}(x) + \frac{\delta S}{\delta\phi(x)}[\hat{\phi}]\right)\hat{\pi}(x)$$

- path integral

$$\langle \phi(x_1)\cdots\phi(x_n)\rangle = \frac{\int [d\pi d\phi] e^{-I_{\mathrm{FP}}}\phi(0,x_1)\cdots\phi(0,x_n)}{\int [d\pi d\phi] e^{-I_{\mathrm{FP}}}}$$

$$I_{\mathrm{FP}} = \int_0^\infty d\tau \left[-\int dx \pi \partial_\tau \phi + H_{\mathrm{FP}}\right]$$

$$
\begin{aligned}
I_{\mathrm{FP}} \quad &= \quad \int_0^\infty d\tau \Big[- \int_0^\infty dL \langle R|\pi_\alpha(\tau, L)\rangle \frac{\partial}{\partial\tau} |\phi^\alpha(\tau, L)\rangle + H(\tau) \\
&\quad + \int_0^\infty dL \Big(\langle R|\mathcal{Q}^\alpha(\tau, L)\rangle |\lambda_\alpha^{\mathcal{Q}}(\tau, L)\rangle + \langle R|\mathcal{T}^\alpha(\tau, L)\rangle |\lambda_\alpha^{\mathcal{T}}(\tau, L)\rangle \Big) \Big]
\end{aligned}
$$

$$
\begin{aligned}
H \quad = \quad & \int_0^\infty dL L \left[\langle R_{12}|\phi^\alpha(L)\rangle_1 |\pi_\alpha(L)\rangle_2 - \langle R_{12}|\pi_\alpha(L)\rangle_1 |\pi_{-\alpha}(L)\rangle_2 \right] \\
& -\frac{1}{2} g_{\mathrm{s}} \int_0^\infty dL_3 \int_0^{L_3} dL_1 \int_0^{L_3 - L_1} dL_2 (L_3 - L_1 - L_2) \\
& \qquad \times \langle G_{0,3,\mathbf{L}}| B^1_{-\alpha_1} B^2_{-\alpha_2} B^3_{\alpha_3} |\phi^{\alpha_1}(L_1)\rangle_1 |\phi^{\alpha_2}(L_2)\rangle_2 |\pi_{\alpha_3}(L_3)\rangle_3 \\
& -g_{\mathrm{s}} \int_0^\infty dL_2 \int_0^\infty dL_3 \int_{|L_2 - L_3|}^{L_2 + L_3} dL_1 (L_2 + L_3 - L_1) \\
& \qquad \times \langle G_{0,3,\mathbf{L}}| B^1_{-\alpha_1} B^2_{\alpha_2} B^3_{\alpha_3} |\phi^{\alpha_1}(L_1)\rangle_1 |\pi_{\alpha_2}(L_2)\rangle_2 |\pi_{\alpha_3}(L_3)\rangle_3 \\
& -g_{\mathrm{s}} \int_0^\infty dL_2 \int_0^\infty dL_3 \int_0^{|L_2 - L_3|} dL_1 \min(L_2, L_3) \\
& \qquad \times \langle G_{0,3,\mathbf{L}}| B^1_{-\alpha_1} B^2_{\alpha_2} B^3_{\alpha_3} |\phi^{\alpha_1}(L_1)\rangle_1 |\pi_{\alpha_2}(L_2)\rangle_2 |\pi_{\alpha_3}(L_3)\rangle_3 ,
\end{aligned}
$$

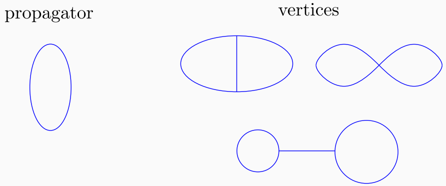
- This action consists of kinetic terms and three string interaction terms.
  - One can calculate the amplitudes perturbatively starting from this action.
  - It is manifestly invariant under the BRST transformation and we can define the physical states using it.

3. Conclusions

5. Conclusions

- We have constructed an SFT for closed bosonic strings based on the Strebel differentials via the Fokker-Planck formalism.

$$I_{\mathrm{FP}} = \int_0^\infty d\tau \Big[- \int_0^\infty dL \langle R|\pi_\alpha(\tau, L)\rangle \frac{\partial}{\partial\tau} |\phi^\alpha(\tau, L)\rangle + H(\tau) + \int_0^\infty dL \Big(\langle R|\mathcal{Q}^\alpha(\tau, L)\rangle |\lambda_\alpha^{\mathcal{Q}}(\tau, L)\rangle + \langle R|\mathcal{T}^\alpha(\tau, L)\rangle |\lambda_\alpha^{\mathcal{T}}(\tau, L)\rangle \Big) \Big]$$



- AdS/CFT?
- Our method may be used to study the classical solutions of the nonpolynomial SFT. (hyperbolic case: Firat-Valdes-Meller)